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# A NUMERICAL SCHEME USING MULTI-SHOCKPEAKONS TO COMPUTE SOLUTIONS OF THE DEGASPERIS-PROCESI EQUATION

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Abstract. We consider a numerical scheme for entropy weak solutions of the DP (Degasperis-Procesi) equation $u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}$. Multi-shockpeakons, functions of the form

$$u(x,t) = \sum_{i=1}^{n} (m_i(t) - \operatorname{sign}(x - x_i(t)) s_i(t)) e^{-|x - x_i(t)|},$$

are solutions of the DP equation with a special property; their evolution in time is described by a dynamical system of ODEs. This property makes multi-shockpeakons relatively easy to simulate numerically. We prove that if we are given a non-negative initial function $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ such that $u_0 - u_{0,x}$ is a positive Radon measure, then one can construct a sequence of multi-shockpeakons which converges to the unique entropy weak solution in $\mathbb{R} \times [0, T)$ for any $T > 0$. From this convergence result, we construct a multi-shockpeakon based numerical scheme for solving the DP equation.

## 1. Introduction

Degasperis and Procesi [11] showed that the Degasperis-Procesi (DP) equation and the Cammassa-Holm (CH) equation are the only two completely integrable equations in the following family of third order nonlinear dispersive PDEs

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad (x,t) \in \mathbb{R} \times [0, \infty), \tag{1.1}$$

for $b \in \mathbb{R}$. The Cauchy problem for the dispersionless CH equation ($b = 2$) is

$$\begin{gathered} u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (x,t) \in \mathbb{R} \times [0, \infty), \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{1.2}$$

In one interpretation it describes finite length small amplitude radial deformation waves in cylindrical compressible hyperelastic rods. Constantin, Escher and Molinet [9, 10] proved that if $u(\cdot, 0) = u_0 \in H^1(\mathbb{R})$ and $u_0 - u_{0,xx}$ is a positive Radon measure, then equation (1.2) has a unique weak solution $u \in C([0,T), H^1(\mathbb{R}))$ for

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any positive $T > 0$. Functions of the form

$$u(x,t) = \sum_{i=1}^{n} m_i(t) e^{-|x-x_i(t)|}, \tag{1.3}$$

called multi-peakons, are weak solutions to the CH equation where the evolution of $m_i$ and $x_i$ is described by the following system of ODEs

$$\dot{x}_i = \sum_{j=1}^{n} m_j e^{-|x_i - x_j|}, \quad \dot{m}_i = m_i \sum_{j=1}^{n} m_j \operatorname{sign}(x_i - x_j) e^{-|x_i - x_j|}.$$

Multi-peakons have been studied and used to study more general solutions of the CH equation by Camassa [1], Camassa, Holm and Hyman [2], Camassa, Huang and Lee [3, 4], and Holden and Raynaud [14]. In particular, Holden and Raynaud proved that if $u_0 \in H^1(\mathbb{R})$ and $u_0 - u_{0,xx}$ is a (positive) Radon measure with a corresponding unique weak solution $u$, then one can construct a sequence of multi-peakons which converges to $u$ in $L^\infty_{\text{loc}}(\mathbb{R}; H^1(\mathbb{R}))$. This result was used to construct a numerical method which approximates solutions of the CH equation with converging sequences of multi-peakons.

The Cauchy problem for the DP equation ($b = 3$) is

$$\begin{gathered} u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad (x,t) \in \mathbb{R} \times [0,T), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{1.4}$$

Many existence, stability and uniqueness results have been achieved for this equation. In [17] Yin proved that if $u_0 \in H^s(\mathbb{R})$ with $s \geq 3$, and $u_0 \in L^3(\mathbb{R})$ is such that $m_0 := u_0 - u_{0,xx} \in L^1(\mathbb{R})$ is non-negative, then equation (1.4) possesses a unique global solution in $C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R}))$. And in [18] Yin showed that if $u_0 \in H^1(\mathbb{R}) \cap L^3(\mathbb{R})$ and $u_0 - u_{0,xx}$ is a non-negative bounded Radon measure on $\mathbb{R}$, then (1.4) has a unique weak solution in $W^{1,\infty}(\mathbb{R} \times \mathbb{R}_+) \cap L^\infty_{\text{loc}}(\mathbb{R}; H^1(\mathbb{R}))$.

Extending the definition of weak solution to entropy weak solution, Coclite and Karlsen [5] showed that if $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then there exists a unique entropy weak solution to (1.4) satisfying $u \in L^\infty([0,T); L^2(\mathbb{R})) \cap L^\infty([0,T); BV(\mathbb{R}))$ for any positive $T$. In [6] they extended uniqueness results by proving uniqueness (but not existence) for entropy weak solutions of (1.4) if $u_0 \in L^\infty(\mathbb{R})$.

Coclite, Karlsen and Risebro [8] investigated various numerical schemes for the entropy weak formulation of the DP equation and provided numerical examples suggesting that discontinuous solutions form independently of the smoothness of the initial data.

Lundmark [15] showed that multi-shockpeakons are entropy weak solutions of the DP equation. Those are functions of the form

$$u(x,t) = \sum_{i=1}^{n} (-\operatorname{sign}(x - x_i(t)) s_i(t) + m_i(t)) e^{-|x-x_i(t)|}, \tag{1.5}$$

whose evolution is determined by the dynamical system of ODEs

$$\dot{x}_i = \sum_{j=1}^{n} \Big( m_j - \operatorname{sign}(x_i - x_j) s_j \Big) e^{-|x_i - x_j|},$$

$$\begin{aligned}\dot{m}_i &= -2m_i \sum_{j=1}^{n} (s_j - \operatorname{sign}(x_i - x_j) m_j) e^{-|x_i - x_j|} \\ &\quad + 2s_i \sum_{j=1}^{n} (m_j - \operatorname{sign}(x_i - x_j) s_j) e^{-|x_i - x_j|}, \\ \dot{s}_i &= -s_i \sum_{j=1}^{n} \Big( s_j - \operatorname{sign}(x_i - x_j) m_j \Big) e^{-|x_i - x_j|}.\end{aligned}$$

Similarly to the multi-peakon results Holden and Raynaud achieved for CH equation, we will show that if we are given an initial function $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ and $u_0 - u_{0,x}$ is a positive Radon measure, then one can construct a sequence of multi-shockpeakons converging to the unique entropy weak solution. To give a better understanding of this initial function space, $\{u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) | u_0 - u_{0,x} \in \mathscr{M}_+(\mathbb{R})\}$, it can also be described as the set of non-negative functions in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$ whose growth is less than exponential almost everywhere, or, as will be done in this paper, as the space

$$\begin{aligned}\big\{ f \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) : \langle f, \phi \rangle \geq 0 \text{ and } \langle f - f_x, \phi \rangle \geq 0 \\ \text{for all non-negative } \phi \in \mathcal{D}(\mathbb{R}) \big\}.\end{aligned}$$

It should be noted that while Yin [18] studied the Cauchy problem (1.4) with the condition $u_0 - u_{0,xx} \in \mathscr{M}_+(\mathbb{R})$, we study the same Cauchy problem with the different condition $u_0 - u_{0,x} \in \mathscr{M}_+(\mathbb{R})$. The respective conditions are both introduced to obtain norm estimates, but for different reasons. The condition $u_0 - u_{0,xx} \in \mathscr{M}_+(\mathbb{R})$ is introduced by Yin to obtain the norm estimates needed to prove existence, stability, and uniqueness of global weak solutions of (1.4). We, however, introduce the condition $u_0 - u_{0,x} \in \mathscr{M}_+(\mathbb{R})$ to obtain the norm estimates needed to prove that the sequence of multi-shockpeakons we use to approximate an entropy weak solution of (1.4) converges strongly to the solution.

The rest of this paper is structured as follows. In Section 2 we give definitions of weak solutions and entropy weak solutions of (1.4), and we include some useful results. Sections 3 and 4 look at properties of multi-shockpeakons and their corresponding dynamical system of ODEs. Section 5 describes a set of entropy weak solutions that can be approximated by multi-shockpeakons. Lastly, Section 6 provides a numerical scheme for solving some Cauchy problems for the DP equation.

## 2. Preliminaries

Applying the operator $(1 - \partial_x^2)^{-1} = \frac{1}{2} e^{-|x|} *$ to equation (1.4), we get the formally equivalent representation of the DP equation

$$u_t + \partial_x \Big[ \frac{1}{2} u^2 + P^u \Big] = 0, \quad P^u = \frac{3}{4} e^{-|x|} * u^2. \tag{2.1}$$

This representation is the basis for the Coclite and Karlsen weak DP solution formulation.

**Definition 2.1** (Weak solution)**.** For a given $T > 0$, we say that $u \in L^\infty(\mathbb{R} \times [0, T))$ is a weak solution of the Cauchy problem (1.4) if the following condition is satisfied

$$\int_0^T \int_{\mathbb{R}} u\phi_t + \frac{1}{2} u^2 \phi_x - P_x^u \phi \, dx dt + \int_{\mathbb{R}} u(x, 0) \phi(x, 0) dx = 0, \tag{2.2}$$

for all $\phi \in \mathcal{D}(\mathbb{R} \times [0,T))$. To achieve stability and uniqueness for DP solutions, Coclite and Karlsen introduced a Kruzkov-type entropy condition.

**Definition 2.2** (Entropy weak solution)**.** A function $u \in L^\infty(\mathbb{R} \times [0,T))$ is an entropy weak solution of the Cauchy problem (1.4) if

(1) $u$ is a weak solution in the sense of Definition 2.1.
(2) For any convex $C^2$ entropy $\eta : \mathbb{R} \to \mathbb{R}$ with corresponding entropy flux $q : \mathbb{R} \to \mathbb{R}$ defined by $q'(u) = \eta'(u)u$, we have that
$$\int_0^T \int_{\mathbb{R}} \eta(u)\phi_t + q(u)\phi_x - \phi\eta'(u)P_x^u dxdt + \int_{\mathbb{R}} \phi(x,0)\eta(u_0(x))dx \geq 0, \tag{2.3}$$
for all $\phi \in \mathcal{D}(\mathbb{R} \times [0,T))$ such that $\phi \geq 0$.

**Remark 2.3.** If we knew that the chain rule held for our weak solutions, we would have uniqueness. But, since our weak solutions include discontinuous solutions for which the chain rule does not hold, the Kruzkov-type entropy condition is imposed to give stability and, thereby, uniqueness.

At the end of this section, we recall two results for $BV(\mathbb{R})$ which will be useful later on.

**Definition 2.4.** A function $f \in L^1(\mathbb{R})$ has bounded variation in $\mathbb{R}$ if
$$\|Df\|(\mathbb{R}) = \sup\Big\{ \int_{\mathbb{R}} f\phi_x dx : \phi \in C_c^1(\mathbb{R}),\, |\phi| \leq 1 \Big\} < \infty.$$
The norm is defined
$$\|f\|_{BV(\mathbb{R})} := \|f\|_{L^1(\mathbb{R})} + \|Df\|(\mathbb{R}),$$
and we write $f \in BV(\mathbb{R})$ if $\|f\|_{BV(\mathbb{R})} < \infty$.

**Theorem 2.5** ([12, Theorem 5-2.1])**.** *Suppose $f_k \in BV(\mathbb{R})$ $(k = 1, \dots)$ and $f_k \to f$ in $L^1_{\rm loc}(\mathbb{R})$. Then*
$$\|Df\|(\mathbb{R}) \leq \liminf_{k\to\infty} \|Df_k\|(\mathbb{R}).$$

## 3. Shockpeakons and multi-shockpeakons

DP entropy weak solutions can be discontinuous. Examples of such solutions are shockpeakons, functions of the form
$$u(x,t) = m_1(t)e^{-|x-x_1(t)|} - s_1(t)\operatorname{sign}(x - x_1(t))e^{-|x-x_1(t)|}. \tag{3.1}$$

The position, $x_1(t)$, describes the position of the shockpeakon, the momentum, $m_1(t)$, describes the strength/"movement" force of the function at the position $x_1(t)$, and the shock, $s_1(t)$, describes a shockpeakon jump discontinuity of $-2s_1(t)$ at the position $x_1(t)$.

The multi-shockpeakon is defined as follows.

**Definition 3.1** (Multi-shockpeakon)**.** Let $G_i$ and $G_i'$ be defined as follows
$$\begin{aligned} G_i = G(x - x_i) &:= e^{-|x-x_i(t)|}, \\ G_i' = G'(x - x_i) &:= -\operatorname{sign}(x - x_i(t))G(x - x_i(t)). \end{aligned} \tag{3.2}$$
A sum of shockpeakons
$$u = \sum_{i=1}^n m_iG_i + s_iG_i', \quad u \in L^\infty(\mathbb{R} \times [0,T))$$

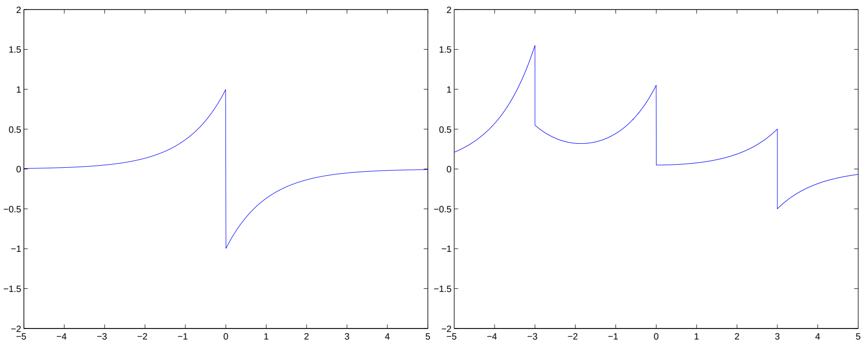


FIGURE 1. Illustrations of a shockpeakon ($x_1 = 0$, $m_1 = 0$ and $s_1 = 1$) (left figure) and a multi-shockpeakon ($\vec{x} = (-3, 0, 3)$, $\vec{m} = (1, 0.5, 0)$ and $\vec{s} = (0.5, 0.5, 0.5)$) (right figure)

is called a multi-shockpeakon. Multi-shockpeakons are sorted by position

$$-\infty < x_1(0) < x_2(0) < \cdots < x_{n-1}(0) < x_n(0) < \infty,$$

and all momenta and shocks are initially bounded

$$|m_i(0)|, |s_i(0)| < \infty, \quad \forall i \in \{1, 2, \dots, n\}.$$

The evolution of the multi-shockpeakon components is described by the dynamical system of ODEs

$$\begin{aligned} \dot{x}_k &= u(x_k), \\ \dot{m}_k &= 2s_k u(x_k) - 2m_k\{u_x(x_k)\}, \\ \dot{s}_k &= -s_k\{u_x(x_k)\}, \end{aligned} \tag{3.3}$$

where

$$u(x_k) = \sum_{i=1}^{n} m_i(t)G(x_k - x_i) + s_i(t)G'(x_k - x_i), \tag{3.4}$$

and the curly brackets denote the nonsingular part

$$\{u_x(x_k)\} := \sum_{i=1}^{n} m_i(t)G'(x_k - x_i) + s_i(t)G(x_k - x_i). \tag{3.5}$$

Furthermore, all shocks are non-negative

$$s_i(t) \geq 0 \quad \forall t \in [0, T), \quad \forall i \in \{1, 2, \dots, n\},$$

that means that all jump discontinuities are downward in $x$.

The simulations of multi-shockpeakons in figure 3 show that shockpeakons can collide. But if we for the time being restrict ourselves to multi-shockpeakons which do not have shockpeakon collisions in the timespan $[0, T)$, then we are able to state the following theorem.

**Theorem 3.2.** *Let $u = \sum_{i=1}^{n} m_i G_i + s_i G_i'$ be a multi-shockpeakon in $L^\infty(\mathbb{R} \times [0, T))$ for which in the timespan $[0, T)$ all positions $x_i$ are distinct in the following sense*

$$x_i(t) \neq x_{i+1}(t), \quad \forall i \in \{1, \dots, n-1\}, \quad t \in [0, T).$$

*Then $u$ is an entropy weak solution.*

*Sketch of proof.* To prove that $u$ is an entropy weak solution, one has to show that $u$ is a weak solution (2.2) and that it fulfills the entropy condition (2.3).

The weak solution part can be proved by inserting $u$ into the weak solution expression. After some computation one will see that the ODEs (3.3) are defined such that $u$ is a weak solution. It is a straightforward, but long computation. For those interested, we refer to the proof by Lundmark in [15, Appendix A].

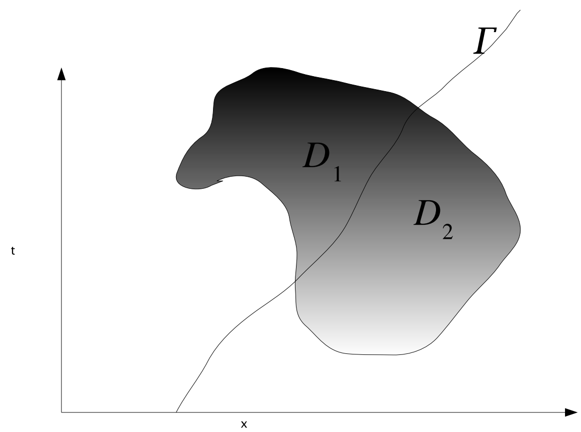


FIGURE 2. Isolated discontinuity curve $\Gamma$ with an open set $D$ intersecting the curve.

Proving the entropy fulfillment, we first notice that multi-shockpeakons are continuous everywhere except on the curves $\Gamma = (x_i(t), t)\big|_{t=0}^{T}$. Let $u_l := u(x_i^-)$ and $u_r := u(x_i^+)$ be the left and right limits of $\Gamma$, respectively and choose an open neighborhood $D \subset \mathbb{R} \times [0, T)$ intersecting the curve $\Gamma$. Assume $D$ is so small that $u$ is smooth in all of $D$ except on $\Gamma$. As depicted in figure 2, let $D_1$ denote the part of $D$ on the left of $\Gamma$ and $D_2$ denote the part on the right. Then, the entropy condition (2.3) implies that for all non-negative test functions $\phi$ we achieve the inequality

$$\begin{aligned}
0 &\le \iint_{D_1\cup D_2} \eta(u)\phi_t + q(u)\phi_x - \eta'(u)\phi P_x^u \,dxdt \\
&= \iint_{D_1\cup D_2} \partial_t(\eta(u)\phi) + \partial_x(q(u)\phi) - \big(\eta(u)_t + q(u)_x + \eta'(u)P_x^u\big)\phi \,dxdt \\
&= \int_{\partial D_1\setminus\partial D} \eta(u)\phi\frac{-\dot{x}_i}{\sqrt{1+\dot{x}_i^2}} + q(u)\phi\frac{1}{\sqrt{1+\dot{x}_i^2}}\,dS \\
&\quad + \int_{\partial D_2\setminus\partial D} \eta(u)\phi\frac{\dot{x}_i}{\sqrt{1+\dot{x}_i^2}} - q(u)\phi\frac{1}{\sqrt{1+\dot{x}_i^2}}\,dS \\
&\quad - \iint_{D_1\cup D_2} \big(u_t + uu_x + P_x^u\big)\eta'(u)\phi\,dxdt, \\
&= \int_\Gamma \Big((\eta(u_r)-\eta(u_l))\frac{\dot{x}_i}{\sqrt{1+\dot{x}_i^2}} - (q(u_r)-q(u_l))\frac{1}{\sqrt{1+\dot{x}_i^2}}\Big)\phi\,dS.
\end{aligned} \tag{3.6}$$

This inequality can be rewritten as

$$q(u_r) - q(u_l) \le \dot{x}_i(\eta(u_r) - \eta(u_l)) \quad \text{on } \Gamma. \tag{3.7}$$

Using that $\dot{x}_i = (u_l + u_r)/2$ and that the entropy condition is satisfied if and only if it is satisfied for all Kruzkov entropies

$$\eta(u) := |u-k|, \quad q(u) := \operatorname{sign}(u-k)\frac{u^2-k^2}{2}, \quad k \in \mathbb{R},$$

we can restate inequality (3.7) as

$$0 \le (u_l - k)|u_r - k| - (u_r - k)|u_l - k|, \quad \text{on } \Gamma, \ \forall k \in \mathbb{R}.$$

Investigating this expression for all $k$ values

$$\begin{aligned} &(u_l - k)|u_r - k| - (u_r - k)|u_l - k| \\ &= \begin{cases} -\Big((u_l-k)(u_r-k) - (u_r-k)(u_l-k)\Big) = 0, & k \ge \max(u_l, u_r) \\ < 0, & u_r > k > u_l \\ > 0, & u_l > k > u_r \\ \Big((u_l-k)(u_r-k) - (u_r-k)(u_l-k)\Big) = 0, & k \le \min(u_l, u_r), \end{cases} \end{aligned}$$

we conclude that multi-shockpeakons satisfy the entropy condition (2.3) if and only if $u_l \ge u_r$, or equivalently; if and only if all shocks $s_i$ are non-negative. ☐

## 4. Shockpeakon collisions

The information provided by the ODEs in (3.3) makes it possible to describe some multi-shockpeakon solutions explicitly. For example, in the single shockpeakon case $u = m_1 G_1 + s_1 G_1'$ with initial conditions $x_1(0), m_1(0) \in \mathbb{R}$ and $s_1(0) \in \mathbb{R}_+$, we get

$$\begin{aligned} \dot{m}_1 &= 0 \implies m_1(t) = m_1(0), \\ \dot{x}_1 &= m_1 \implies x_1(t) = x_1(0) + m_1(0)t, \\ \dot{s}_1 &= -s_1^2 \implies s_1(t) = s_1(0)/(1 + ts_1(0)). \end{aligned}$$

Which means that the shockpeakon moves at the constant speed $m_1(0)$ and its shock decreases until it reaches zero;

$$u(x,t) = m_1(0)G\big(x - (x_1(0) + m_1(0)t)\big) + \frac{s_1(0)}{1 + ts_1(0)}G'\big(x - (x_1(0) + m_1(0)t)\big).$$

In [16] Lundmark and Szmigielski used an inverse scattering approach to find explicit solutions for all shockless multi-shockpeakons, called multi-peakons, with strictly positive/negative momenta. For the general multi-shockpeakon, however, we do not have an explicit solution, but since the ODEs (3.3) implicitly describe multi-shockpeakon evolution, we can find numerical algorithms to approximate the explicit solution. In fact, such approximations will converge to the explicit solution as long as all components $x_i$, $m_i$, and $s_i$ are bounded in the solution domain. However, collisions can occur between shockpeakons, and as a result of that, momenta can blow up. For example[1], if $u = m_1 G_1 + m_2 G_2$ with initial conditions

$$0 < -x_1(0) = x_2(0) \quad \text{and} \quad 0 < m_1(0) = -m_2(0),$$

then a collision will occur at the time $\tilde{t} = x_2(0)/\big(m_1(0)(1 - e^{-2x_2(0)})\big)$. And at that time momenta blow up; $\lim_{t\to\tilde{t}^-} m_1(t) = \lim_{t\to\tilde{t}^-} -m_2(t) = \infty$.

[1] See [15] for a more detailed explanation of this example and shockpeakon collisions in general.

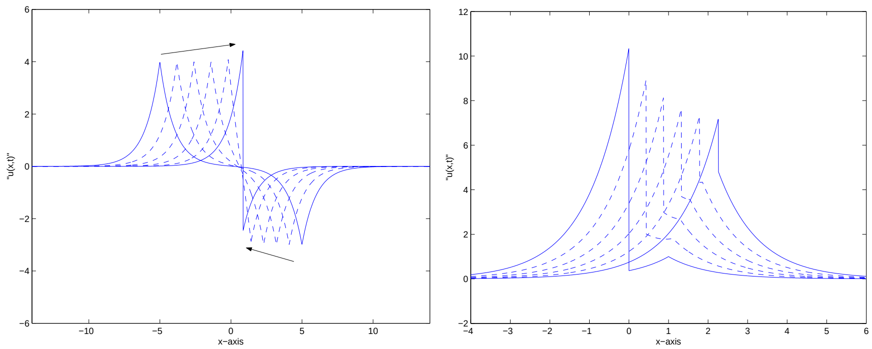


FIGURE 3. Examples of shockpeakon collisions. In the left figure one shockpeakons with positive momentum collides with one shockpeakon with negative momentum. Momenta blow up, this results in a shock. In the right figure, one shockpeakon with positive momentum and shock collides with one shockpeakon with positive momentum. At the collision, the shockpeakons fuse into one shockpeakon. Momenta stay bounded.

It turns out that some shockpeakon collisions result in momenta blow-ups and some do not (see figure 3). When trying to determine the evolution of a multi-shockpeakon it is difficult to continue the multi-shockpeakon past collisions with momenta blow-ups in such a way that it remains an entropy weak solution. For collisions where momenta stay bounded, however, it is not difficult to determine multi-shockpeakon evolution even at collisions. Summing up, we wished to approximate multi-shockpeakon entropy weak solutions numerically based on the ODEs (3.3). But, since shockpeakon collision treatment is too difficult when momenta blow up, we restrict ourselves to looking at a set of multi-shockpeakons for which momenta stay bounded at all times. For multi-shockpeakons in this set, we will describe the way to treat shockpeakon collisions such that these multi-shockpeakons are entropy weak solutions in $\mathbb{R}\times[0,T)$ for any given $T>0$. But first, let us include a useful conservation result for multi-shockpeakons.

**Proposition 4.1.** *A multi-shockpeakon $u=\sum_{i=1}^n m_iG_i+s_iG_i'$, conserves momentum. That is, $\sum_{i=1}^n \dot m_i=0$.*

*Proof.* We recall from equations (3.3), (3.4) and (3.5) that

$$\dot m_i = 2s_iu(x_i)-2m_i\{u_x(x_i)\},$$

where

$$u(x_i)=\sum_{j=1}^n m_jG(x_i-x_j)+s_jG'(x_i-x_j),$$

$$\{u_x(x_j)\}:=\sum_{j=1}^n m_jG'(x_i-x_j)+s_jG(x_i-x_j).$$

Furthermore, recall from equation (3.2) that $G(x_i-x_j)=G(x_j-x_i)$ and $G'(x_i-x_j)=-G'(x_i-x_j)$. We use these properties to prove that momentum is conserved by a straightforward calculation.

$$
\begin{aligned}
\sum_{i=1}^{n} \dot{m}_i &= 2\sum_{i=1}^{n} \Big(s_i u(x_i) - m_i\{u_x(x_i)\}\Big) \\
&= 2\sum_{i=1}^{n} s_i\Big(\sum_{j=1}^{n} m_j G(x_i - x_j) + s_j G'(x_i - x_j)\Big) \\
&\quad - 2\sum_{i=1}^{n} m_i\Big(\sum_{j=1}^{n} m_j G'(x_i - x_j) + s_j G(x_i - x_j)\Big) \\
&= 2\sum_{i,j=1}^{n} \Big((s_i m_j - m_i s_j) G(x_i - x_j) + (s_i s_j - m_i m_j) G'(x_i - x_j)\Big) \\
&= 0.
\end{aligned}
$$

□

**Theorem 4.2.** *Let $\mathscr{F}$ be the set of entropy weak multi-shockpeakons with shocks initially dominated by the corresponding momenta*

$$
\mathscr{F} = \Big\{ \sum_{i=1}^{n} m_i G_i + s_i G_i' : m_i(0) \geq s_i(0) \geq 0 \ \textit{and}\ M = \sum_{i=1}^{n} m_i(0) < \infty \Big\}.
$$

*If $u \in \mathscr{F}$, then:*

(1) *Shocks will be dominated by corresponding momenta in the function domain:*

$$
s_i(t) \leq m_i(t), \quad \forall t \in \mathbb{R}_+,\ i \in \{1, \dots, n\}.
$$

(2) *All the components of $u$ will be bounded and thereby Lipschitz continuous in the function domain with Lipschitz constants as follows, for all $i \in \{1, \dots, n\}$,*

$$
\begin{gathered}
|\dot{x}_i| = |u(x_i)| \leq 2M \\
|\dot{m}_i| = |2 s_i u(x_i) - 2 m_i\{u_x(x_i)\}| \leq 8M^2 \\
|\dot{s}_i| = |-s_i\{u_x(x_i)\}| \leq 2M^2
\end{gathered}
\tag{4.1}
$$

(3) *If collisions between two or more shockpeakons occur, then the colliding shockpeakons $\{(x_i, m_i, s_i)\}_{i=i_1}^{i_2}$ fuse into the single shockpeakon*

$$
\begin{gathered}
\hat{x} = x_{i_1} = x_{i_1+1} = \dots = x_{i_2}, \\
\hat{m} = \sum_{i=i_1}^{i_2} m_i \quad \textit{and} \quad \hat{s} = \sum_{i=i_1}^{i_2} s_i.
\end{gathered}
\tag{4.2}
$$

*Proof.* (1) Assume that there is a time $\tilde{t} \in \mathbb{R}_+$, such that $m_j(\tilde{t}) = s_j(\tilde{t})$ for one or more $j \in \{1, 2, \dots, n\}$. Using the ODEs (3.3), we see that

$$\begin{aligned} \dot{m}_j - \dot{s}_j &= 2s_j u(x_j) - 2m_j\{u_x(x_j)\} + s_j\{u_x(x_j)\} \\ &= s_j(2u(x_j) - \{u_x(x_j)\}) \\ &= s_j\Big(\sum_{i=1}^{n}(2m_i - s_i)G(x_j - x_i) + (2s_i - m_i)G'(x_j - x_i)\Big) \\ &\geq s_j\Big(\sum_{i=1}^{j-1}(m_i - s_i)G(x_j - x_i) + m_j + 2\sum_{i=j+1}^{n} s_i G(x_j - x_i)\Big) \\ &\geq 0. \end{aligned} \tag{4.3}$$

At the time $\tilde{t}$ we have $\dot{m}_j(\tilde{t}) - \dot{s}_j(\tilde{t}) \geq 0$. Hence, $s_j$ will always be less or equal to $m_j$.

(2) From part 1 and the conservation of momentum we deduce that

$$S = \sum_{i=1}^{n} s_i \leq \sum_{i=1}^{n} m_i = M,$$

and thus, for all $(x, t) \in \mathbb{R} \times \mathbb{R}_+$,

$$\begin{gathered} 0 \leq u(x,t) \leq M + S \leq 2M, \\ |\{u_x(x,t)\}| = |\sum_{i=1}^{n} m_i(t)G'(x_k - x_i) + s_i(t)G(x_k - x_i)| \leq 2M\,. \end{gathered}$$

By inserting these inequalities into the ODEs (3.3), we find the Lipschitz constants (4.1).

(3) The $x_i$, $m_i$, and $s_i$ components are Lipschitz continuous. This implies that if a collision occurs between shockpeakons $\{(x_i, m_i, s_i)\}_{i=i_1}^{i_2}$ at time $\tilde{t}$, none of the involved components will jump in value. Consequently, the continuation of the solution beyond a collision consists of all shockpeakons not involved in the collision, and the shockpeakon created by the collision described in (4.2). We conclude that given

$$u(x,t) = \sum_{i=1}^{n} m_i(t)G(x - x_i(t)) + s_i(t)G'(x - x_i(t)), \quad 0 \leq t < \tilde{t},$$

the continuation beyond the collision time becomes

$$\begin{aligned} u(x,t) = \sum_{i=1}^{i_1-1} &\Big(m_i(t)G(x - x_i(t)) + s_i(t)G'(x - x_i(t))\Big) \\ &+ \hat{m}(t)G(x - \hat{x}(t)) + \hat{s}(t)G'(x - \hat{x}(t)) \\ &+ \sum_{i=i_2+1}^{n} m_i(t)G(x - x_i(t)) + s_i(t)G'(x - x_i(t)), \end{aligned}$$

for $\tilde{t} \leq t$ less than the next collision time. ∎

## 5. Multi-shockpeakon approximation results

In this section we will show that multi-shockpeakons in $\mathscr{F}$ are entropy weak solutions. Thereafter, we will find the class of DP weak entropy solutions which are approximable by multi-shockpeakons from $\mathscr{F}$. But first, let us include a well-posedness result for entropy weak DP solutions obtained by Coclite and Karlsen.

**Theorem 5.1** ([5, Theorem 3.1]). *Suppose $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then there exists an entropy weak solution to the Cauchy problem*

$$\begin{gathered} u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \\ u(x,0) = u_0(x). \end{gathered} \tag{5.1}$$

*Fix any $T > 0$, and let $u, v : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be two entropy weak solutions of (5.1) with initial data $u_0, v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, respectively. Then for almost all $t \in [0, T)$,*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \le e^{M_T t} \|u_0 - v_0\|_{L^1(\mathbb{R})}, \tag{5.2}$$

*where*

$$M_T := \frac{3}{2}\Big(\|u\|_{L^\infty(\mathbb{R}\times[0,T))} + \|v\|_{L^\infty(\mathbb{R}\times[0,T))}\Big) < \infty.$$

*Consequently, there exists at most one entropy weak solution to (5.1). The solution satisfies the following estimate for almost all $t \in [0, T)$*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})} + 24t\|u_0\|_{L^2(\mathbb{R})}. \tag{5.3}$$

**Remark 5.2.** Originally, in [5], the wellposedness result above was given for initial functions $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. However, from private conversations with Coclite and Karlsen we were informed that it is possible to extend uniqueness to initial functions in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ [7].

The theorem above tells us that to show that multi-shockpeakons in $\mathscr{F}$ are entropy weak solutions, we need boundedness results for $\mathscr{F}$.

**Theorem 5.3.** *If*

$$\sum_{i=1}^n m_i G_i + s_i G_i' = u \in \mathscr{F} \quad \text{and} \quad M := \sum_{i=1}^n m_i(t),$$

*then*

1. $u \ge 0$.
2. $|u(x,t)| \le 2M$, *for all* $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.
3. $\|u(\cdot,t)\|_{L^p(\mathbb{R})} \le (2M)^p$, *for all* $p \in [1,\infty)$, $t \in \mathbb{R}_+$.
4. $\|u(\cdot,t)\|_{BV(\mathbb{R})} \le 8M$, *for all* $t \in \mathbb{R}_+$.
5. $\|u(\cdot,t) - u(\cdot,\tau)\|_{L^1(\mathbb{R})} \le (32M^3 + 20M^2)|t-\tau| + O(|t-\tau|^2)$, *for all* $(t,\tau) \in \mathbb{R}_+^2$.

*Proof.* (1) Use the property $m_i \ge s_i$ from theorem 4.2 to deduce that

$$u = \sum_{i=1}^n m_i G_i + s_i G_i' \ge \sum_{i=1}^n (m_i - s_i) G_i \ge 0.$$

(2) Note that

$$u(x,t) = \sum_{i=1}^n m_i G_i + s_i G_i' \le 2M.$$

(3) First, we prove this statement for $p = 1$.

$$\|u(\cdot,t)\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \sum_{i=1}^{n} m_i(t)G_i + s_i(t)G_i' dx = 2M. \tag{5.4}$$

For $p \in (1,\infty)$, we use (5.4) and statement (2),

$$\|u(\cdot,t)\|_{L^p(\mathbb{R})}^p \le \|u(\cdot,t)\|_{L^\infty(\mathbb{R})}^{p-1} \|u(\cdot,t)\|_{L^1(\mathbb{R})} \le (2M)^p. \tag{5.5}$$

(4) Using the definition of bounded variation, definition 2.4, yields

$$\begin{aligned} \|u(\cdot,t)\|_{BV(\mathbb{R})} &\le \int_{\mathbb{R}} |u| + |u_x| dx \\ &= 2M + \int_{\mathbb{R}} \Big| \sum_{i=1}^{n} m_i G_i' + s_i G_i + 2 s_i \delta_{x_i} \Big| dx \\ &\le 4\|u(\cdot,0)\|_{L^1(\mathbb{R})}. \end{aligned} \tag{5.6}$$

(5) See Hoel [13, Theorem 5.2]. □

With these boundedness results we will be able to prove that multi-shockpeakons in $\mathscr{F}$ are entropy weak solutions. But first, let us introduce some notation for the set of initial functions corresponding to $\mathscr{F}$. Let $T_0 : \mathscr{F} \to L^\infty(\mathbb{R})$ be the mapping defined by $T_0(u) = u(\cdot,0)$ and define the set of multi-shockpeakon initial functions $\mathscr{F}(0)$ by $\mathscr{F}(0) := T_0(\mathscr{F})$. That is,

$$u_0 \in \mathscr{F}(0) \iff \begin{cases} u_0(x) = \sum_{i=1}^{n} m_i(0)G(x - x_i(0)) + s_i(0)G'(x - x_i(0)), \\ m_i(0) \ge s_i(0) \ \ \forall i, \text{ and } \sum_{i=1}^{n} m_i(0) < \infty. \end{cases}$$

**Theorem 5.4.** *If $u_0 \in \mathscr{F}(0)$, then its multi-shockpeakon continuation $u$ is the entropy weak solution to the Cauchy problem (5.1) for any $T > 0$.*

*Proof.* The boundedness criterion $u \in L^\infty(\mathbb{R} \times [0,T))$ follows from theorem 5.3 (2). When proving that the weak solution condition (2.2) is fulfilled, collisions are an obstacle. Theorem 3.2 proves that multi-shockpeakons whose shockpeakons are position-wise distinct ($x_i \neq x_{i+1} \quad \forall i$) in the timespan $[0,T)$ are weak entropy solutions in $\mathbb{R} \times [0,T)$. However, at shockpeakon collisions, some shockpeakons do, by definition, share position ($x_i = x_{i+1}$). But the way multi-shockpeakon solutions of $\mathscr{F}$ are continued past collisions (see theorem 4.2) ensures that all shockpeakons are locally distinct; it is only at collision times that some of the shockpeakons share position.

If a multi-shockpeakon initially has $n$ shockpeakons, then we know from the fact that at a collision two or more shockpeakons fuse into one shockpeakon that there is at most $n$ times for which the entropy condition is not satisfied.

Thus for any multi-shockpeakon $u = \sum_{i=1}^{n} m_i G_i + s_i G_i' \in \mathscr{F}$ with $k$ ($0 \le k \le n$) shockpeakon collisions at the times $\{t_k\}_{i=1}^{k} \subset (0,T)$, theorem 3.2 says that $u$ is a weak solution in the spaces $\mathbb{R} \times [0,t_1)$, $\mathbb{R} \times (t_i, t_{i+1})$ $\forall i \in \{1,\dots,k-1\}$, and $\mathbb{R} \times (t_k, T)$. This means that for all test functions $\phi \in \mathcal{D}(\mathbb{R} \times [0,T))$ we have that for all $i \in \{2,\dots,k-1\}$,

$$\int_{t_i^+}^{t_{i+1}^-} \int_{\mathbb{R}} u\phi_t + \frac{1}{2}u^2\phi_x - P_x^u \phi dx dt + \int_{\mathbb{R}} u(x,t_i^+)\phi(x,t_i) - u(x,t_{i+1}^-)\phi(x,t_{i+1}) dx = 0$$

Using this property and theorem 5.3 (5), we deduce that the weak solution criterion (2.2) holds for $u$ by the following calculation

$$
\begin{aligned}
0 &= \int_0^{t_1}\int_{\mathbb{R}} u\phi_t + \frac{1}{2}u^2\phi_x - P_x^u\phi dxdt + \int_{\mathbb{R}} u(x,0)\phi(x,0)dx \\
&\quad + \sum_{i=1}^{k-1}\Big[\int_{t_i^+}^{t_{i+1}^-}\int_{\mathbb{R}} u\phi_t + \frac{1}{2}u^2\phi_x - P^{u_x}\phi dxdt \\
&\quad + \int_{\mathbb{R}} u(x,t_i^+)\phi(x,t_i) - u(x,t_{i+1}^-)\phi(x,t_{i+1})dx\Big] \\
&\quad + \int_{\mathbb{R}} u(x,t_k^+)\phi(x,t_k)dx + \int_{t_k^+}^{T}\int_{\mathbb{R}} u\phi_t + \frac{1}{2}u^2\phi_x - P_x^u\phi dxdt \\
&= \int_0^{T}\int_{\mathbb{R}} u\phi_t + \frac{1}{2}u^2\phi_x - P_x^u\phi dxdt + \int_{\mathbb{R}} u(x,0)\phi(x,0)dx \\
&\quad + \sum_{i=1}^{k}\int_{\mathbb{R}} \phi(x,t_i)(u(x,t_i^+) - u(x,t_i^-))dx \\
&= \int_0^{T}\int_{\mathbb{R}} u\phi_t + \frac{1}{2}u^2\phi_x - P_x^u\phi dxdt + \int_{\mathbb{R}} u(x,0)\phi(x,0)dx \\
&\quad \forall\phi \in \mathcal{D}(\mathbb{R}\times[0,T)).
\end{aligned}
\tag{5.7}
$$

An argument of the same type proves that the entropy condition holds. □

From theorem 5.4 we know that if $u_0 \in \mathscr{F}(0)$, then its entropy weak solution $u$ lies in $\mathscr{F}$. So an entropy weak solution of the Cauchy problem (5.1) with $u_0 \in \mathscr{F}(0)$ is trivially (perfectly) approximable by multi-shockpeakons from $\mathscr{F}$. However, we will show that the space of $\mathscr{F}$ approximable solutions is bigger. Let $\langle\cdot,\cdot\rangle : \mathcal{D}'(\mathbb{R})\times\mathcal{D}(\mathbb{R}) \to \mathbb{R}$ be $\langle f,\phi\rangle = \int_{\mathbb{R}} f(x)\phi(x)dx$ and define the set of initial functions

$$
\begin{aligned}
\mathscr{H} = \big\{ &f \in L^1(\mathbb{R})\cap BV(\mathbb{R}) : \langle f,\phi\rangle \ge 0 \text{ and } \langle f - f_x,\phi\rangle \ge 0 \\
&\text{for all non-negative } \phi \in \mathcal{D}(\mathbb{R})\big\}.
\end{aligned}
$$

**Theorem 5.5** ($\mathscr{F}$-approximable entropy weak DP solutions)**.** *Suppose $u_0$ belongs to $L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, then there exists a sequence of multi-shockpeakons $u_n \in \mathscr{F}$ which converges to the entropy weak solution $u$ in $L^p(\mathbb{R}\times[0,T))$ for all $p \in [1,\infty)$, $T > 0$ if and only if $u_0 \in \mathscr{H}$.*

*Proof.* The proof of this theorem is divided into the following three propositions:
Proposition 5.6: Proving the existence of a sequence of functions $\{u_{0,n}\}_n \subset \mathscr{F}(0)$ converging to $u_0$ in $L^1$.
Proposition 5.7: Proving that the multi-shockpeakon continuation series of $\{u_{0,n}\}_n$, $\{u_n\}_n \subset \mathscr{F}$, converges to the entropy weak solution $u$.
Proposition 5.8: Proving the only if statement; if there for an entropy weak solution $u$ with initial function $u_0 \in L^1\cap L^\infty$ exists a sequence of multi-shockpeakons in $\mathscr{F}$ which converges to $u$ in $L^p(\mathbb{R}\times[0,T))$ $\forall p \in [1,\infty)$, then $u_0 \in \mathscr{H}$. □

**Proposition 5.6.** *Suppose $u_0 \in \mathscr{H}$, then there exists a sequence $\{u_{0,n}\} \subset \mathscr{F}(0)$ such that*

$$
\begin{aligned}
u_{0,n} &\to u_0, \quad \text{in } L^1(\mathbb{R}), \\
u_{0,n} &\le u_0, \quad \forall n \in \mathbb{N}.
\end{aligned}
\tag{5.8}
$$

*Proof.* We begin by mollifying $u_0$. Let

$$\rho(x) := \begin{cases} Ce^{\frac{1}{x^2-1}}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases}$$

with the constant $C$ chosen so that $\int_{\mathbb{R}} \rho(x)\, dx = 1$. Define $\rho_\varepsilon(x) = (1/\varepsilon)\rho(x/\varepsilon)$, and mollify $u_0$ by $u_0^\varepsilon = \rho_\varepsilon * u_0$. The mollified function $u_0^\varepsilon$ converges to $u_0$ in $L^1$ as $\varepsilon$ goes to zero. Furthermore, $u_0^\varepsilon$ is smooth, non-negative and

$$(u_0^\varepsilon)_x = \rho_\varepsilon * u_0'(x) \leq \rho_\varepsilon * u_0(x) = u_0^\varepsilon(x). \tag{5.9}$$

In particular, (5.9) implies

$$u_0^\varepsilon(x) \geq u^\varepsilon(y)e^{x-y}, \quad \forall x < y. \tag{5.10}$$

The convergence is proved in two steps. First, we show that for each $\varepsilon > 0$ we can find a sequence of functions $\{u_{0,\varepsilon,n}\}_n \subset \mathscr{F}(0)$ converging to $u_0^\varepsilon$ in $L^1$. Second, we pick a sequence $\{u_{0,k}\}_k \subset \{u_{0,\varepsilon,n}\}_{\varepsilon,n}$ that converges to $u_0$ in $L^1$.

Choose the sequence $\{u_{0,\varepsilon,n}\} \subset \mathscr{F}(0)$ with shocks equal to momenta:

$$u_{0,\varepsilon,n} = \sum_{i=1}^{n^2} m_i G_i(x - x_i) + m_i G_i'(x - x_i).$$

We determine the shockpeakon values $\{x_i, m_i, m_i\}_{i=1}^{n^2}$ in $u_{0,\varepsilon,n}$ starting with the last one

$$\begin{gathered} x_i = \frac{2n(i-1)}{n^2-1} - n, \quad i \in \{1, 2, \dots, n^2\}, \\ s_{n^2} = m_{n^2} = \frac{u_0^\varepsilon(x_{n^2})}{2}, \\ s_i = m_i = \frac{u_0^\varepsilon(x_i) - \sum_{j=i+1}^{n^2} m_j(G(x_j - x_i) + G'(x_j - x_i))}{2}, \\ i \in \{n^2-1, n^2-2, \dots, 1\}. \end{gathered} \tag{5.11}$$

One thing needs justification in the determination scheme; shocks must be non-negative. The last shock, $s_{n^2}$, is non-negative since $u_0^\varepsilon$ is non-negative. Using (5.10) we find that

$$\begin{aligned} &u_0^\varepsilon(x) - m_{n^2}(G(x - x_{n^2}) + G'(x - x_{n^2})) \\ &= \begin{cases} u_0^\varepsilon(x) \geq 0, & x > x_{n^2}, \\ \frac{u_0^\varepsilon(x_{n^2})}{2} \geq 0, & x = x_{n^2}, \\ u_0^\varepsilon(x) - u_0^\varepsilon(x_{n^2}^-)e^{x - x_{n^2}^-} \geq 0, & x < x_{n^2}. \end{cases} \end{aligned} \tag{5.12}$$

Hence

$$2s_{n^2-1} = u_0^\varepsilon(x_{n^2-1}) - m_{n^2}(G(x_{n^2-1} - x_{n^2}) + G'(x_{n^2-1} - x_{n^2})) \geq 0.$$

Define

$$u_{0,\varepsilon,n}^1 := u_0^\varepsilon \chi_{(-\infty, x_{n^2}]} - m_{n^2}(G_{n^2} + G_{n^2}').$$

Then $u_{0,\varepsilon,n}^1$ is non-negative everywhere and smooth on $(-\infty, x_{n^2})$. Similarly to (5.12) we obtain

$$u_{0,\varepsilon,n}^1 - m_{n^2-1}(G_{n^2-1} + G_{n^2-1}') \geq 0.$$

We see that $s_{n^2-2}$ is non-negative by the inequality

$$u_0^\varepsilon - \sum_{j=n^2+1-2}^{n} m_j(Gj + G'j) = u_0^\varepsilon \chi_{(x_{n^2},\infty)} + u^1_{0,\varepsilon,n} - m_{n^2-1}(G_{n^2-1} + G'_{n^2-1}) \geq 0.$$

Inductively, we construct functions

$$u^i_{0,\varepsilon,n} := u^{i-1}_{0,\varepsilon,n}\chi_{(-\infty,x_{n^2+1-i}]} - m_{n^2+1-i}(G_{n^2+1-i} + G'_{n^2+1-i}) \quad i = 2,\dots,n^2 \tag{5.13}$$

such that each $u^i_{0,\varepsilon,n}$ is non-negative everywhere and smooth on $(-\infty, x_{n^2+1-i})$, and use these to verify that shocks are non-negative

$$\begin{aligned} 2s_{n^2-i} &= u_0^\varepsilon - \sum_{j=n^2+1-i}^{n^2} m_j(G_j + G'_j) \\ &\geq u^i_{0,\varepsilon,n} \geq 0 \quad \forall i \in \{1,2,\dots,n^2-1\}. \end{aligned}$$

By (5.13) we can also show that $(u_{0,\varepsilon,n})_{n\in\mathbb{N}}$ is bounded by $u_0^\varepsilon$;

$$u_0^\varepsilon - u_{0,\varepsilon,n} \geq u^{n^2}_{0,\varepsilon,n} \geq 0 \quad \forall n \in \mathbb{N}. \tag{5.14}$$

The convergence $u_{0,\varepsilon,n} \to u_0^\varepsilon$ is proved as follows. Since $u_0^\varepsilon \in L^1(\mathbb{R})$ we know that for any $\epsilon > 0$, there exists an $R(\epsilon) \in \mathbb{R}_+$ such that

$$\|u_0^\varepsilon(1 - \chi_{(-r,r)})\|_{L^1(\mathbb{R})} \leq \frac{\epsilon}{2}, \quad \text{if } r \geq R(\epsilon).$$

Let $\lfloor x \rfloor \in \mathbb{Z}$ be the integer part of $x \in \mathbb{R}$ and notice that

$$\sup_{x\in(x_i,x_{i+1})} u_0^\varepsilon(x_{i+1}) - u_{0,\varepsilon,n}(x) \leq u_0^\varepsilon(x_{i+1})(1 - e^{x_i - x_{i+1}}).$$

Then

$$\begin{aligned} &\|(u_0^\varepsilon - u_{0,\varepsilon,n})\chi_{(-r,r)}\|_{L^1(\mathbb{R})} \\ &\leq \sum_{i=\lfloor\frac{(n-r)(n^2-1)}{2n}\rfloor}^{\lfloor\frac{(r+n)(n^2-1)}{2n}\rfloor} \int_{(x_i,x_{i+1})} u_0^\varepsilon(x) - u_{0,\varepsilon,n}(x)dx \\ &\leq \sum_{i=\lfloor\frac{(n-r)(n^2-1)}{2n}\rfloor}^{\lfloor\frac{(r+n)(n^2-1)}{2n}\rfloor} \int_{(x_i,x_{i+1})} \|u_0^\varepsilon\|_{BV((x_i,x_{i+1}))} + u_0^\varepsilon(x_{i+1}) - u_{0,\varepsilon,n}(x)dx \\ &\leq \|u_0^\varepsilon\|_{BV(\mathbb{R})}\frac{2n}{n^2-1} + (r + \frac{n}{n^2-1})(1 - e^{-\frac{2n}{n^2-1}})\|u_0^\varepsilon\|_{L^\infty(\mathbb{R})} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Consequently, for any $\epsilon > 0$ we can find an $R(\epsilon) \in \mathbb{R}$ and an $N(r,\epsilon) \in \mathbb{N}$ such that

$$\begin{aligned} \|(u_0^\varepsilon - u_{0,\varepsilon,n})\|_{L^1(\mathbb{R})} &\leq \|u_0^\varepsilon(1 - \chi_{(-r,r)})\|_{L^1(\mathbb{R})} + \|(u_0^\varepsilon - u_{0,\varepsilon,n})\chi_{(-r,r)}\|_{L^1(\mathbb{R})} \\ &\leq \epsilon, \quad \text{if } r \geq R(\epsilon) \text{ and } n \geq N(r,\epsilon). \end{aligned}$$

To prove general convergence, let $\varepsilon_n = 1/n$,

$$h(n) := \min\big\{m \in \mathbb{N} \big| \|u_0^{\varepsilon_n} - u_{0,\varepsilon_n,m}\|_{L^1(\mathbb{R})} \leq \frac{1}{n}\big\}$$

and define the sequence $\{u_{0,n}\}_n$ by $u_{0,n} = u_{0,\varepsilon_n,h(n)}$. Then, for any $\epsilon > 0$ there exist a natural number $N \in \mathbb{N}$ such that

$$\|u_0 - u_0^{\varepsilon_n}\|_{L^1(\mathbb{R})} \le \frac{\epsilon}{2} \quad \text{and} \quad \|u_0^{\varepsilon_n} - u_{0,n}\|_{L^1(\mathbb{R})} \le \frac{\epsilon}{2}, \quad \text{if } n \ge N.$$

Consequently,

$$\|u_0 - u_{0,n}\|_{L^1(\mathbb{R})} \le \|u_0 - u_0^{\varepsilon_n}\|_{L^1(\mathbb{R})} + \|u_0^{\varepsilon_n} - u_{0,n}\|_{L^1(\mathbb{R})} \le \epsilon, \quad \text{if } n \ge N.$$

□

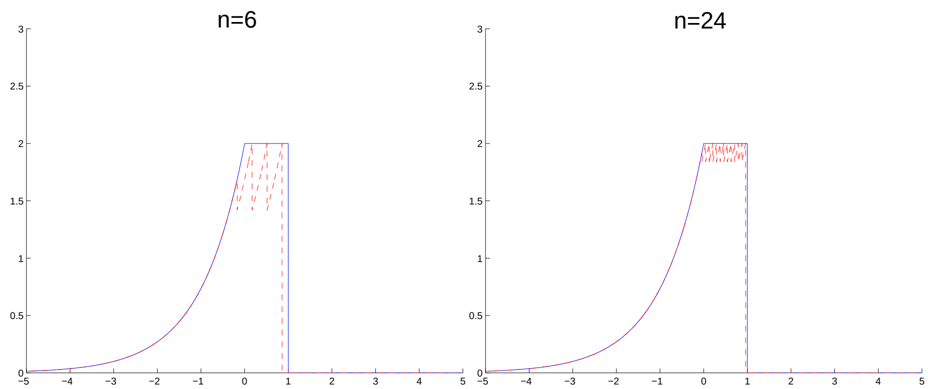


FIGURE 4. Making $\{u_{0,n}\}$ from (5.8) converge to $u_0(x) = 2e^x\chi_{(-\infty,0]}(x) + 2\chi_{(0,1]}(x)$ by using the scheme in proposition 5.6.

**Proposition 5.7.** *Given $T > 0$ and $u_0 \in \mathscr{H}$ with a corresponding entropy weak solution $u$. Assume that $\{u_{0,n}\} \subset \mathscr{F}(0)$ is a sequence converging to $u_0$ in $L^1$. Then the corresponding entropy solutions of $\{u_{0,n}\}$, $\{u_n\} \subset \mathscr{F}$, converge to $u$ in the following sense*

$$u_n \to u \quad \text{in } L^p(\mathbb{R} \times [0,T)), \quad \forall p \in [1,\infty). \tag{5.15}$$

*Proof.* Since $u_0 \in \mathscr{H} \subset L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ theorem 5.1 states that for almost all $t \in [0,T)$

$$\|u(\cdot,t) - u_n(\cdot,t)\|_{L^1(\mathbb{R})} \le e^{M_T^n t}\|u_0 - u_{0,n}\|_{L^1(\mathbb{R})},$$

where

$$M_T^n = \frac{3}{2}\Big(\|u\|_{L^\infty(\mathbb{R}\times[0,T))} + \|u_n\|_{L^\infty(\mathbb{R}\times[0,T))}\Big).$$

From theorem 5.3 (ii) and the boundedness property $u_{0,n} \le u_0$ we obtain an $L^\infty$ bound on $\{u_n\}_n$:

$$\|u_n\|_{L^\infty(\mathbb{R}\times[0,T))} \le \|u_{0,n}\|_{L^1(\mathbb{R})} \le \|u_0\|_{L^1(\mathbb{R})} \quad \forall n \in \mathbb{N}.$$

Combing this bound with the $L^\infty$ bound on $u$ in (5.3), gives a global bound on all $M_T^n$ constants

$$M_T^n \le \|u_0\|_{L^1(\mathbb{R})} + \|u_0\|_{L^\infty(\mathbb{R})} + 24T\|u_0\|_{L^2(\mathbb{R})}, \quad \forall n \in \mathbb{N}.$$

Thus, for almost all $t \in [0,T)$

$$\|u(\cdot,t) - u_n(\cdot,t)\|_{L^1(\mathbb{R})} \le e^{(\|u_0\|_{L^1(\mathbb{R})} + \|u_0\|_{L^\infty(\mathbb{R})} + 24T\|u_0\|_{L^2(\mathbb{R})})t}\|u_0 - u_{0,n}\|_{L^1(\mathbb{R})}.$$

Since $u_{0,n} \to u_0$ in $L^1$, $u \in L^\infty(\mathbb{R} \times [0,T))$ and $\{u_0\}_n \subset L^\infty(\mathbb{R} \times [0,T))$, the convergence described in (5.15) is obtained. □

So far we have shown that if an initial function $u_0$ lies in $\mathscr{H}$, then its corresponding unique entropy solution is approximable by multi-shockpeakons from $\mathscr{F}$. Next, we will show that $\mathscr{H} = \overline{\mathscr{F}(0)}^{L^1(\mathbb{R})}$. This implies that for an entropy weak solution $u$ with $u(\cdot,0) = u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ there exists a sequence of multi-shockpeakons in $\mathscr{F}$ which converges to $u$ in $L^p(\mathbb{R} \times [0,T))$ $\forall p \in [1,\infty)$ only if $u_0 \in \mathscr{H}$.

**Proposition 5.8.** *The closure of $\mathscr{F}(0)$ in $L^1$ fulfills the equality*

$$\overline{\mathscr{F}(0)}^{L^1(\mathbb{R})} = \mathscr{H}.$$

*Proof.* We know from proposition 5.6 that $\mathscr{H} \subseteq \overline{\mathscr{F}(0)}^{L^1(\mathbb{R})}$. The proposition will be proved by verifying the opposite inclusion $\mathscr{H} \supseteq \overline{\mathscr{F}(0)}^{L^1(\mathbb{R})}$.

If $u_0 \in \overline{\mathscr{F}(0)}^{L^1(\mathbb{R})}$, then there exists a sequence $\{u_{0,n}\}_n \subset \mathscr{F}(0)$ such that $u_{0,n}$ converge to $u_0$ in $L^1$ by definition. Furthermore,

(1) $u_0 \in BV(\mathbb{R})$. Which can be proved as follows: For any $\varepsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that

$$\|u_{0,n}\|_{L^1(\mathbb{R})} \le \|u_0\|_{L^1(\mathbb{R})} + \varepsilon, \quad \forall n > N_1.$$

Theorem 2.5 and equation (5.6) give us that

$$\|Du_0\|(\mathbb{R}) \le \liminf_{n\to\infty} \|Du_{0,n}\|(\mathbb{R}) \le 3\|u_0\|_{L^1(\mathbb{R})}.$$

Using definition 2.4, we end up with the proposed norm

$$\|u_0\|_{BV(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})} + \|Du_0\|(\mathbb{R}) \le 4\|u_0\|_{L^1(\mathbb{R})}.$$

(2) $\langle u_0, \phi\rangle \ge 0$ for all non-negative $\phi \in \mathcal{D}(\mathbb{R})$. Proof: Since all functions in $\mathscr{F}(0)$ are non-negative, we get

$$\langle u_0, \phi\rangle \ge \int_{\mathbb{R}} (u_0 - u_{0,n})\phi\, dx \to 0 \quad \text{as } n \to \infty.$$

(3) $\langle u_0 - \frac{d}{dx}u_0, \phi\rangle \ge 0$ for all non-negative $\phi \in \mathcal{D}(\mathbb{R})$. Proof: Observe that

$$u_{0,n} - \frac{d}{dx}u_{0,n} = \sum_{i=1}^{n^2} (m_i - s_i)(G_i - G_i') + 2s_i\delta_{x_i} \ge 0.$$

Thus for all non-negative $\phi \in \mathcal{D}(\mathbb{R})$ we have

$$\begin{aligned}\langle u_0 - \frac{d}{dx}u_0, \phi\rangle &\ge \int_{\mathbb{R}} \Big(u_0 - \frac{d}{dx}u_0 - (u_{0,n} - \frac{d}{dx}u_{0,n})\Big)\phi\, dx \\ &= \int_{\mathbb{R}} (u_0 - u_{0,n})(\phi + \phi_x)\, dx \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

So if $u_0 \in \overline{\mathscr{F}(0)}^{L^1(\mathbb{R})}$, then

$$\begin{aligned}u_0 \in \big\{f \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) : \langle f, \phi\rangle \ge 0 \text{ and } \langle f - f_x, \phi\rangle \ge 0 \\ \text{for all non-negative } \phi \in \mathcal{D}(\mathbb{R})\big\}.\end{aligned}$$

In other words, $\overline{\mathscr{F}(0)}^{L^1(\mathbb{R})} \subseteq \mathscr{H}$. □

**A related approximation result.** If we look at the space consisting of negative momenta and shock components

$$\mathscr{F}^- := \Big\{u = \sum_{i=1}^{n} m_i G_i + s_i G_i' : m_i(0) \le -s_i(0) \le 0 \text{ and } M = \sum_{i=1}^{n} m_i(0) > -\infty\Big\},$$

we achieve, by a computation similar to (4.3), that

$$m_i(t) \le -s_i(t) \quad \forall t \in \mathbb{R}_+.$$

With this key property we can show that $\mathscr{F}^-$ has analogous properties to those $\mathscr{F}$ has. We define

$$\begin{aligned}\mathscr{H}^- = \big\{ & f \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) : \langle f, \phi\rangle \le 0 \text{ and } \langle f - f_x, \phi\rangle \le 0 \\ & \text{for all non-negative } \phi \in \mathcal{D}(\mathbb{R})\big\},\end{aligned}$$

and state the following theorem.

**Theorem 5.9.** *Suppose $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then there exist a sequence of multi-shockpeakons $u_n \in \mathscr{F}^-$ which converges to the entropy weak solution $u$ in $L^p(\mathbb{R} \times [0, T))$ for all $p \in [1, \infty)$, $T > 0$ if and only if $u_0 \in \mathscr{H}^-$.*

The proof of this theorem is similar to the proof of theorem 5.5.

## 6. Numerical simulations

In the numerical simulations we use multi-shockpeakons from $\mathscr{F}$ to approximate entropy weak solutions with initial data in $\mathscr{H}$. For a given initial function, the algorithm of theorem 5.6 (equation (5.11)) is used to determine initial momentum and shock values of multi-shockpeakons $u_n$ (for $n = 3, 6, 12, 24, 48$). Thereafter we use Matlab with the explicit Runge-Kutta solver ODE45 to solve the ODEs (3.3) for the $u_n$ functions. Assuming $u_{\text{exact}}$ is the exact solution, error is evaluated by taking the $L^1$ norm[2] of $u_n - u_{\text{exact}}$ at the following times $t = (0, 2, 5)$. However, except for $t = 0$ we do not know the exact solution for most of the initial functions we are studying. When the exact solution is unknown, we approximate it with a high resolution $u_n$ function; $u_{\text{exact}} := u_{48}$ for $t > 0$, ($u_{48}$ consists of up to $48^2$ shockpeakons).

It is difficult to simulate shockpeakon collisions well numerically. Due to the usage of discrete time steps, shockpeakons who are supposed to collide can move past each other. But we want shockpeakons to fuse at collisions, just like they do in the continuum setting. To make sure this happens we implement a discrete collision fusing operation in our simulations. Choose an $\epsilon > 0$ such that two shockpeakons are set to be one - by manipulating shockpeakon positions - if their distance is less than $\epsilon$ (this is not completely accurate, see the fusing operation in the box below for a more precise description of when shockpeakons are fused together). Let $\Delta t \le 2\epsilon/\|u_0\|_{L^1(\mathbb{R})}$ be the time step, and for each simulation time $k\Delta t$, $k \in \{1, 2, \dots\}$ perform the following fusing operation

$$\boxed{\begin{array}{l}\text{for } i = n-1, n-2 \dots, 1 \\ \qquad \text{If } |x_{i+1}(k\Delta t) - x_i(k\Delta)| \le \epsilon \\ \qquad\qquad \text{set } x_i(k\Delta t) = x_{i+1}(k\Delta t)\end{array}}$$

[2] We compute the $L^1$ norm by numerical integration of $|u_n - u_{\text{exact}}|$ restricted to $(-100, 100)$. This is sufficiently correct for the initial functions we are looking at.

The accuracy of collision treatment in the simulations increase as $\epsilon$ decreases. In the examples we use $\epsilon = 10^{-4}$ and $\Delta t \le 2\epsilon/5$.

**Example 6.1.** We look at the peakon function

$$u_0(x) = 2e^{-|x|}. \tag{6.1}$$

By the ODEs (3.3) we find the explicit DP solution corresponding to this initial function

$$u_{\text{exact}}(x,t) = 2e^{-|x-2t|}.$$

The simulations indicate that our approximations obtain/maintain a peakon structure just like the exact solution. But for low $n$-values the approximate solutions $u_n$ move at slower speeds than the exact solution. This is because the shockpeakons we are using to approximate the exact solution are asymmetric. The convergence rate is close to linear and, unfortunately, errors grow in time.

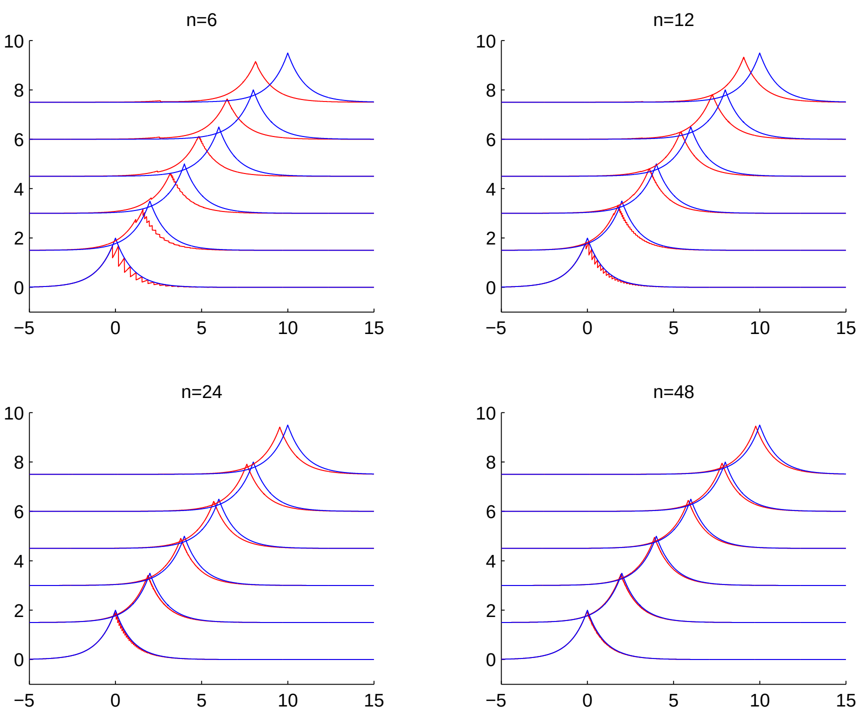


FIGURE 5. Simulation of the solution $u_{\text{exact}}$ (in blue) and approximate solutions (in red) for example 6.1 at the times $t = (0,1,2,3,4,5)$.

**Example 6.2.** The second function we look at is

$$u_0(x) = 2e^{x}\chi_{(-\infty,0]}(x) + 2\chi_{(0,1]}(x). \tag{6.2}$$

A multi-shockpeakon approximation to this initial function is illustrated in figure 4. The simulation $u_{\text{exact}}$ indicates that the shape of this function transforms into a peakon. But this should not be overemphasized; all our numerical simulations

| $n$-value | 3 | 6 | 12 | 24 | 48 |
|---:|---:|---:|---:|---:|---:|
| $\|u_n - u_{\text{exact}}\|_{L^1(\mathbb{R})}$ at $t=0$ | 1.10 | 0.63 | 0.35 | 0.16 | 0.08 |
| Ratio | 0 | 1.74 | 1.81 | 2.14 | 1.97 |
| $\|u_n - u_{\text{exact}}\|_{L^1(\mathbb{R})}$ at $t=2$ | 3.45 | 2.55 | 1.46 | 0.78 | 0.40 |
| Ratio | 0 | 1.35 | 1.75 | 1.87 | 1.94 |
| $\|u_n - u_{\text{exact}}\|_{L^1(\mathbb{R})}$ at $t=5$ | 5.35 | 4.53 | 2.88 | 1.63 | 0.86 |
| Ratio | 0 | 1.18 | 1.58 | 1.77 | 1.88 |

TABLE 1. Error estimates for approximate solutions of (6.1).

consist of shockpeakons, therefore it seems highly probable that they will transform from whatever initial shape into a set of dispersed peakons/shockpeakons. Furthermore, the error estimates are similar to those in the first example although this example uses an approximated exact solution.

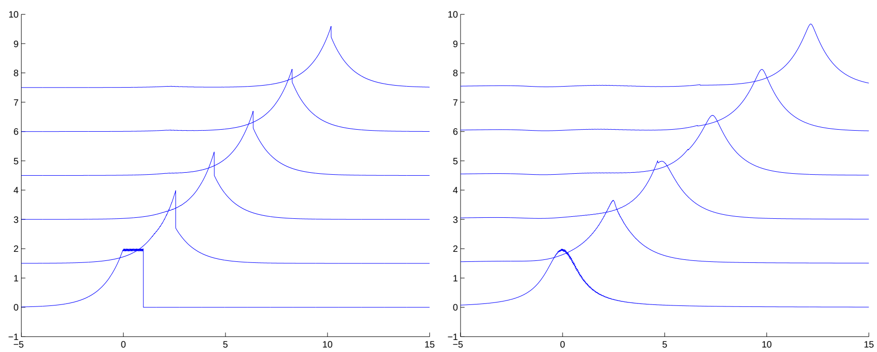

FIGURE 6. Simulations of $u_{\text{exact}}$ at the times $t = (0, 1, 2, 3, 4, 5)$ for example 6.2 (left figure) and example 6.3 (right figure).

| $n$-value | 3 | 6 | 12 | 24 | 48 |
|---:|---:|---:|---:|---:|---:|
| $\|u_n - u_{\text{exact}}\|_{L^1(\mathbb{R})}$ at $t=0$ | 0.94 | 0.57 | 0.28 | 0.15 | 0.08 |
| Ratio | | 1.65 | 2.04 | 1.85 | |
| $\|u_n - u_{\text{exact}}\|_{L^1(\mathbb{R})}$ at $t=2$ | 3.13 | 2.14 | 1.13 | 0.42 | |
| Ratio | | 1.46 | 1.90 | 2.68 | |
| $\|u_n - u_{\text{exact}}\|_{L^1(\mathbb{R})}$ at $t=5$ | 4.85 | 3.69 | 2.06 | 0.81 | |
| Ratio | | 1.31 | 1.79 | 2.53 | |

TABLE 2. Error estimates for approximate solutions of (6.2).

**Example 6.3.** In the last example we look at the bell shaped function

$$u_0(x) = \frac{2}{1+x^2}. \tag{6.3}$$

As we see from the error estimates, this function is more difficult to approximate by the numerical method than the other ones we have studied. The reason is that, compared to the other functions, it is decaying very slowly as $x \to \infty$.

| $n$-value | 3 | 6 | 12 | 24 | 48 |
|---|---|---|---|---|---|
| $\|u_n - u_{\text{exact}}\|_{L^1(\mathbb{R})}$ at $t = 0$ | 2.53 | 1.44 | 0.76 | 0.39 | 0.17 |
| Ratio | | 1.76 | 1.91 | 1.94 | |
| $\|u_n - u_{\text{exact}}\|_{L^1(\mathbb{R})}$ at $t = 2$ | 5.48 | 3.33 | 1.61 | 0.57 | |
| Ratio | | 1.65 | 2.07 | 2.85 | |
| $\|u_n - u_{\text{exact}}\|_{L^1(\mathbb{R})}$ at $t = 5$ | 7.47 | 5.72 | 3.21 | 1.21 | |
| Ratio | | 1.31 | 1.78 | 2.64 | |

Table 3. Error estimates for approximate solutions of (6.3).

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