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Constructible DP solutions

Master thesis

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Preface

This thesis studies solutions of the nonlinear Degasperis-Procesi equation. In particular, it explores which Degasperis-Procesi solutions are constructible by a set of elementary solutions called multi-shockpeakons. The prerequisites needed for reading this text are few; all results are achieved by basic measure theory and functional analysis. Although I am satisfied with my achievements, does it seem likely that it is possible to prove constructibility for a larger set of DP solutions than I have been able to.

I would like to thank everyone who has helped me writing this thesis, especially my supervisor, Nils Henrik Risebro.

Contents

Chapter 1

Introduction

Degasperis and Procesi showed that the Degasperis-Procesi and Cammassa-Holm equation are the only two completely integrable[1] equations in the following family of third order nonlinear dispersive PDEs

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \qquad (x,t) \in \mathbb{R} \times [0,\infty), \tag{1.1}$$

running over $b \in \mathbb{R}$.

The dispersionless Cammassa-Holm equation ($b = 2$) is

$$\begin{aligned} u_t - u_{xxt} + 3uu_x &= 2u_xu_{xx} + uu_{xxx}, \qquad (x,t) \in \mathbb{R} \times [0,\infty), \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{1.2}$$

In one interpretation, it describes finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods.

Constantin, Escher and Molinet [1, 2] proved that if $u_0 \in H^1(\mathbb{R})$ and $u_0 - u_{0,xx}$ is a positive Radon measure, then equation (1.2) has a unique global weak solution $u \in C([0,T); H^1(\mathbb{R}))$ for any positive T. But it was also shown that solutions with odd initial data u_0 in $H^3(\mathbb{R})$ such that $u_{0,x} < 0$ do blow up in finite time [1].
Functions of the form

$$u(x,t) = \sum_{i=1}^{n} m_i(t) e^{-|x - x_i(t)|}, \tag{1.3}$$

called multi-peakons, are weak solutions to the CH equation with well described evolution. Holden and Raynaud [3] proved that if $u_0 \in H^1((R))$ and $m_0 := u_0 - u_{0,xx}$ is a positive Radon measure, then one can construct a sequence of multipeakons which converges in $L^\infty_{\text{loc}}(\mathbb{R}; H^1(\mathbb{R}))$ to the unique solution u.

The Degasperis-Procesi equation ($b = 3$)

$$\begin{aligned} u_t - u_{txx} + 4uu_x &= 3u_xu_{xx} + uu_{xxx}, \qquad (x,t) \in \mathbb{R} \times [0,\infty), \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R} \end{aligned} \tag{1.4}$$

can be regarded as a model for nonlinear shallow water dynamics. Many existence, stability and uniqueness results have been proved for this equation. For example:

[1] Complete integrability means that there exist a Lax pair formulation of the equation. This means in particular that solutions of such equations satisfy infinitely many conservation laws.

- In [4], Yin proved that if $u_0 \in H^s(\mathbb{R})$ with $s \geq 3$, and $u_0 \in L^3(\mathbb{R})$ is such that $m_0 := u_0 - u_{0,xx} \in L^1(\mathbb{R})$ is non-negative, then equation (1.4) possesses a unique global solution in $C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R}))$.

- In [5], Yin showed that if $u_0 \in H^1(\mathbb{R}) \cap L^3(\mathbb{R})$ and $u_0 - u_{0,xx}$ is a nonnegative bounded Radon measure on $\mathbb{R}$, then (1.4) has a unique weak solution in $W^{1,\infty}(\mathbb{R} \times \mathbb{R}_+) \cap L^\infty_{\text{loc}}(\mathbb{R}; H^1(\mathbb{R}))$.

- Extending the definition of weak solution to entropy weak solution, Coclite and Karlsen [6] showed that if $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then there exists a unique entropy weak solution to (1.4) satisfying $u \in L^\infty([0,T); L^2(\mathbb{R})) \cap L^\infty([0,T); BV(\mathbb{R}))$ for any positive T.

Multi-shockpeakons, functions of the form

$$u^n(x,t) = \sum_{i=1}^{n} (-\text{sign}\,(x - x_i(t))\, s_i(t) + m_i(t)) e^{-|x - x_i(t)|}, \tag{1.5}$$

$$s_i \geq 0 \quad \forall i \in \{1, 2, \ldots, n\} \tag{1.6}$$

are weak entropy solutions of the DP equation. As we mentioned above, Holden and Raynaud showed that some solutions of the CH equation are multi-peakon constructible. This thesis aims to explore if something similar is possible for entropy weak solutions of the DP equation. That is; which entropy weak solutions of the DP equation are constructible by multi-shockpeakons.

Chapter 2

Classical weak solution and peakons

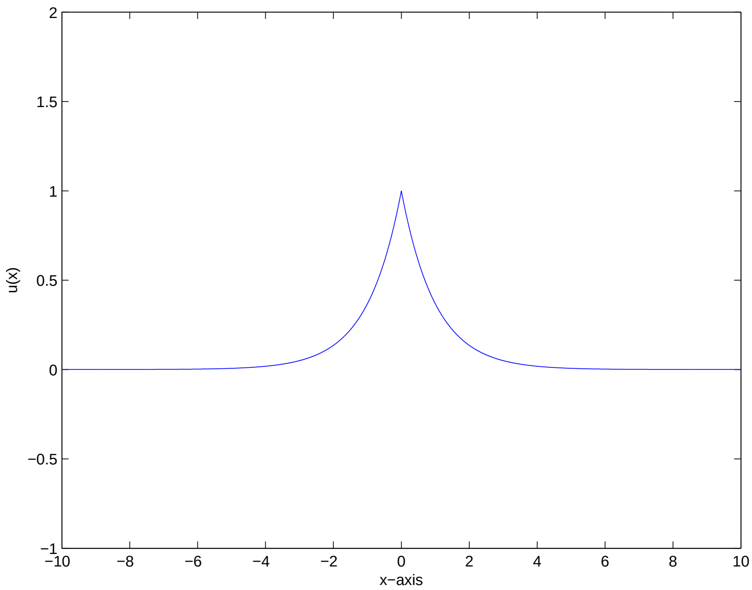

Figure 2.1: Illustration of a peakon $(x(0) = 0, m(0) = 1)$

A peakon (peaked soliton) is a wave function of the form $u = m(t)G(x - x(t))$, where $G(x - x(t)) = e^{-|x-x(t)|}$ and $m(t) > 0$. Anti-peakons are defined similarly, the only difference is that its momentum is negative; $m(t) < 0$. The peakon momentum, $m(t)$, describes the peakon's strength/height at a given time. $x(t)$ describes its position. The derivatives of $m(t)$ and $x(t)$ are determined by the differential equation the peakon solves. For example, if u is a single peakon solving the Degasperis-Procesi equation and we are given initial values $(x_1(0), m_1(0))$, then

$$\dot{m}_1 = 0, \qquad \dot{x}_1 = m_1(t) = m_1(0). \tag{2.1}$$

In this case $u(x,t) = u(x - m_1(0)t, 0)$ is a travelling wave function.

Multi-peakons are weak solutions of the family of equations (1.1) defined the following way:

Definition 2.1 (Multi-peakon). A sum of peakons and anti-peakons of the form

$$u(x,t) = \sum_{i=1}^{n} m_i(t) G(x - x_i(t)), \quad u \in L^{\infty}(\mathbb{R} \times [0,T)), \tag{2.2}$$

is called a multi-peakon. By convention, the sum is always sorted position-wisely

$$-\infty < x_1(0) < x_2(0) < .... < x_{n-1}(0) < x_n(0) < \infty, \tag{2.3}$$

all momenta are initially bounded

$$|m_i(0)| < \infty, \quad \forall i \in \{1, 2, ..., n\}, \tag{2.4}$$

and all positions and momenta are differentiable on $[0, T)$:

$$m_i(t), x_i(t) \in C^1([0,T)), \quad i \in \{1, 2, ..., n\}. \tag{2.5}$$

2.1 Classical weak solution

Rewrite equation (1.1)

$$u_t - u_{xxt} + \frac{b+1}{2}(u^2)_x + \frac{3-b}{2}(u_x^2)_x - \frac{1}{2}(u^2)_{xxx} = 0. \tag{2.6}$$

Definition 2.2. $u \in L^{\infty}([0,T); H^1_{\mathrm{loc}}(\mathbb{R}))$ is weak solution of (2.6) if

$$\int_0^T \int_{\mathbb{R}} \Big((D_t - D^3_{xxt})u + (\frac{b+1}{2} D_x - \frac{1}{2} D^3_{xxx}) u^2 + \frac{3-b}{2} D_x(u_x^2) \Big) \phi \quad dxdt = 0, \tag{2.7}$$

$$\forall \phi \in C_c^{\infty}(\mathbb{R} \times [0,T)). \tag{2.8}$$

Here D is the distributional derivative operator, and we assume that
$u(-\infty, t) = u(\infty, t) = 0, \quad \forall t \geq 0.$

Inspired by Holden and Raynaud [3] and Lundmark [7], I will show that the momenta, $m_k(t)$, and positions, $x_k(t)$, of multi-peakon weak solutions of (2.6) have to satisfy spesific ODEs.

Theorem 2.3. *The multi-peakon*

$$u(x,t) = \sum_{i=1}^{n} m_i(t) G(x - x_i(t)), \tag{2.9}$$

is a weak solution (up to peakon anti-peakon collision (neglect at first read)) of the family of equations (2.6) *if and only if*

$$\dot{x}_k = \sum_{i=1}^{n} m_i(t) G(x_k - x_i), \tag{2.10}$$

$$\dot{m}_k = (b-1) m_k \sum_{i=1}^{n} \mathrm{sign}(x_k - x_i) m_i G(x_k - x_i) \tag{2.11}$$

$$\textit{where } \mathrm{sign}(0) := 0. \tag{2.12}$$

Proof. The boundedness criterion $u \in L^\infty([0,T); H^1_{loc}(\mathbb{R}))$ is most easily proven after having built up notation. So we start by doing that while proving that u satisfies the weak solution criterion (2.7).
To do this we need to compute distributional derivatives $D_x u_x^2$, $D_t u$, $D_{xxt} u$, $D_x u^2$ and $D_{xxx} u^2$. That means we have to differentiate functions of the form $m_1(t) f(x - x_1(t))$. It is proven in Appendix A that general functions of the type $f(x - x_1(t))$ where $x_1(t) \in C^1$, satisfy these chain rules in distributional sense

$$\begin{aligned} D_t f(x - x_1(t)) &= -\dot{x}_1(t) D_{x - x_1(t)} f(x - x_1(t)) \\ &= -\dot{x}_1 f', \\ \\ D_x f(x - x_1(t)) &= D_{x - x_1(t)} f(x - x_1(t)) \\ &= f'. \end{aligned} \tag{2.13}$$

Furthermore, if $m_1(t) \in C^1$, then $D_t(m_1(t) f(x - x_1(t))) = f\dot{m}_1 + \dot{x}_1 f'$.

Studying $D_x u_x^2$, the first problem we encounter is that u_x and hence u_x^2 is not everywhere defined. Differentiating a peakon $u = m_1(t) G(x - x_1(t))$

$$u_x(x,t) = \begin{cases} -\operatorname{sign}(x - x_1(t)) m_1(t) G(x - x_1(t)), & \text{if } x \neq x_1(t) \\ \text{not defined}, & \text{if } x = x_1(t) \end{cases} \tag{2.14}$$

we see that $u_x(x,t)$, for a given t, has x-discontinuities at the $x_i(t)$. Apart from those points $u_x(x,t)$ is smooth. Since $u_x(\cdot, t)$ is not defined everywhere, $D_x u_x^2$ does not, in a sense, make sense. But $u_x(\cdot, t)$ is defined everywhere except a set of measure zero, so we can assign finite values to this set without affecting the distributional derivative. We normalize u_x by taking the average of left and right limits

$$u_x(x_i) := \frac{u_x(x_i^+) + u_x(x_i^-)}{2}, \tag{2.15}$$

By proposition A.5 we know that if g is another function that is smooth almost everywhere and normalized like u_x, then the partial distribution derivative D_x satisfies Leibniz rule $D_x(u_x g) = g D_x u_x + u_x D_x g$.

The normalized u_x reads

$$\begin{aligned} u_x &= \sum_{i=1}^{n} -\operatorname{sign}(x - x_i) m_i G(x - x_i) \\ &= \sum_{i=1}^{n} m_i G'(x - x_i) \end{aligned} \tag{2.16}$$

Here $G'(x - x_i)$ is the normalized derivative of $G(x - x_i)$

$$G'(x - x_i) := -\operatorname{sign}(x - x_i) G(x - x_i) = \begin{cases} -e^{x_i - x} & \text{if } x > x_i \\ 0 & \text{if } x = x_i \\ e^{x - x_i} & \text{if } x < x_i \end{cases} \tag{2.17}$$

Now, u_x satisfies the Lebniz rule, so we have that $D_x(u_x^2) = 2 u_x D_x u_x$. $D_x u_x$ is given by

$$\begin{aligned} D_x u_x &= D_x\Big(\sum_{i=1}^{n} m_i G'(x - x_i)\Big) \\ &= \sum_{i=1}^{n} m_i D_x G'(x - x_i) \end{aligned} \tag{2.18}$$

The function $G'(x-x_i)$ consists of two Leibniz functions, $-\operatorname{sign}(x-x_i)$ and $G(x-x_i)$. The distributional chain rule holds for both. Thus

$$\begin{aligned} D_x G'(x-x_i) &= G(x-x_i)D_x(-\operatorname{sign}(x-x_i)) - \operatorname{sign}(x-x_i)D_x G(x-x_i) \\ &= G(x-x_i)(-2\delta(x-x_i) + (-\operatorname{sign}(x-x_i))^2) \\ &= -2\delta_{x_i} + G(x-x_i). \end{aligned} \tag{2.19}$$

Using this in (2.18) we end up with

$$D_x u_x = \sum_{i=1}^{n} m_i(G(x-x_i) - 2\delta_{x_i}), \tag{2.20}$$

and thereof

$$\begin{aligned} D_x u_x^2 &= 2u_x D u_x \\ &= 2\Big(\sum_{j=1}^{n} m_j G'(x-x_j)\Big)\Big(\sum_{i=1}^{n} m_i(G(x-x_i) - 2\delta_{x_i})\Big) \\ &= 2\sum_{i,j=1}^{n} m_i m_j G'(x-x_j)(G(x-x_i) - 2\delta_{x_i}) \\ &= 2\sum_{i,j=1}^{n} m_i m_j \Big(G'(x-x_j)G(x-x_i) - 2G(x_i-x_j)\delta_{x_i}\Big). \end{aligned} \tag{2.21}$$

We differentiate the other distributional components in the same fashion

$$\begin{aligned} D_t u &= D_t\Big(\sum_{i=1}^{n} m_i G(x-x_i)\Big) \\ &= \sum_{i=1}^{n} \dot{m}_i G(x-x_i) + m_i D_t G(x-x_i) \\ &= \sum_{i=1}^{n} \dot{m}_i G(x-x_i) - m_i \dot{x}_i G'(x-x_i), \end{aligned} \tag{2.22}$$

$$\begin{aligned} D_{xxt} u &= D_{xxt}\Big(\sum_{i=1}^{n} m_i G(x-x_i)\Big) \\ &= D_{xt}\Big(\sum_{i=1}^{n} m_i G'(x-x_i)\Big) \\ &= D_t\Big(\sum_{i=1}^{n} m_i(G(x-x_i) - 2\delta_{x_i})\Big) \\ &= \sum_{i=1}^{n} \dot{m}_i(G(x-x_i) - 2\delta_{x_i}) - \dot{x}_i m_i(G'(x-x_i) - 2\delta'_{x_i}), \end{aligned} \tag{2.23}$$

$$\begin{aligned} D_x u^2 &= 2u D_x u \\ &= 2\sum_{i,j=1}^{n} m_i m_j G'(x-x_i)G(x-x_j), \end{aligned} \tag{2.24}$$

$$
\begin{aligned}
D_{xxx}u^2 &= 2D_{xx}(uD_xu) \\
&= 2D_x(uD_{xx}u + (D_xu)^2) \\
&= 2D_x \sum_{i,j=1}^{n} m_im_j\Big(G(x-x_j)(G(x-x_i)-2\delta_{x_i}) + G'(x-x_i)G'(x-x_j)\Big) \\
&= 2D_x \sum_{i,j=1}^{n} m_im_j\Big(G(x-x_j)G(x-x_i) - 2G(x_i-x_j)\delta_{x_i} + G'(x-x_i)G'(x-x_j)\Big) \\
&= 2\sum_{i,j=1}^{n} m_im_j(4G'(x-x_i)G(x-x_j) - 4G(x_i-x_j)\delta_{x_i} - 2G(x_i-x_j)\delta'_{x_i}).
\end{aligned} \tag{2.25}
$$

Having all the distributional components needed, we insert them into equation (2.6)

$$
\begin{aligned}
0 &= D_tu - D_{xxt}u + \frac{b+1}{2}D_x(u^2) + \frac{3-b}{2}D_x(u_x^2) - \frac{1}{2}D_{xxx}(u^2) \\
&= \sum_{i=1}^{n} \dot{m}_iG(x-x_i) - m_i\dot{x}_iG'(x-x_i) \\
&\quad - \Big(\sum_{i=1}^{n} \dot{m}_i(G(x-x_i) - 2\delta_{x_i}) - \dot{x}_im_i(G'(x-x_i) - 2\delta'_{x_i})\Big) \\
&\quad + \frac{b+1}{2}\Big(2\sum_{i,j=1}^{n} m_im_jG'(x-x_i)G(x-x_j)\Big) \\
&\quad + \frac{3-b}{2}\Big(2\sum_{i,j=1}^{n} m_im_j\big(G'(x-x_j)G(x-x_i) - 2G'(x_i-x_j)\delta_{x_i}\big)\Big) \\
&\quad - \frac{1}{2}\Big(2\sum_{i,j=1}^{n} m_im_j\big(4G'(x-x_i)G(x-x_j) - 4G(x_i-x_j)\delta_{x_i} - 2G(x_i-x_j)\delta'_{x_i}\big)\Big) \\
&= \sum_{i=1}^{n} 2\dot{m}_i\delta_{x_i} - 2m_i\dot{x}_i\delta'_{x_i} \\
&\quad + \sum_{i,j=1}^{n} m_im_j\Big(2(b-1)G'(x_i-x_j)\delta_{x_i} + 2G(x_i-x_j)\delta'_{x_i}\Big) \\
&= \sum_{i=1}^{n} 2\delta_{x_i}\Big(\sum_{j=1}^{n}((b-1)m_im_jG'(x_i-x_j) + \dot{m}_i)\Big) + \sum_{i=1}^{n} 2\delta'_{x_i}\Big(m_i\sum_{j=1}^{n} m_jG(x_i-x_j) - \dot{x}_i\Big).
\end{aligned} \tag{2.26}
$$

Assuming x_i are distinct, the set $(\cup_{i=1}^{n}\delta_{x_i}) \cup (\cup_{i=1}^{n}\delta'_{x_i})$ is linearly independent. Thus for (2.26) to be true, all factors of δ_{x_i} and δ'_{x_i} terms have to be zero. This yields exactly what the theorem states:

$$
\left.
\begin{aligned}
\dot{x}_i &= \sum_{j=1}^{1} m_jG(x_i-x_j) \\
\dot{m}_i &= -(b-1)m_i\sum_{j=1}^{n} m_jG'(x_i-x_j) \\
&= (b-1)m_i\sum_{j=1}^{n} m_j\,\mathrm{sign}(x_i-x_j)G(x_i-x_j)
\end{aligned}
\right\} \text{for } i = \{1, 2, ..., n\}. \tag{2.27}
$$

When proving the boundedness criterion $u \in L^\infty([0,T); H^1_{\text{loc}}(\mathbb{R}))$, equations will get ugly unless we compress notation. Introduce

$$G_i := G(x - x_i), \qquad G_i' := G'(x - x_i). \tag{2.28}$$

$$\begin{aligned} \|u(\cdot,t)\|^2_{H^1_{loc}(\mathbb{R})} &\leq \int_{\mathbb{R}} u^2 + (D_x u)^2 dx \\ &= \int_{\mathbb{R}} \sum_{i,j=1}^{n} m_i m_j (G_i G_j + G_i' G_j') dx \end{aligned} \tag{2.29}$$

Calculating integrands gives

$$\begin{aligned} \int_{\mathbb{R}} G_i G_j dx &= \int_{-\infty}^{min(x_i,x_j)} e^{2x - x_i - x_j} dx + \int_{min(x_i,x_j)}^{max(x_i,x_j)} e^{min(x_i,x_j) - max(x_i,x_j)} dx \\ &\quad + \int_{max(x_i,x_j)}^{\infty} e^{x_i + x_j - 2x} dx \\ &= e^{min(x_i,x_j) - max(x_i,x_j)} \big(1 + (max(x_i,x_j) - min(x_i,x_j))\big) \\ &= e^{-|x_i - x_j|} \big(1 + |x_i - x_j|\big), \end{aligned} \tag{2.30}$$

$$\begin{aligned} \int_{\mathbb{R}} G_i' G_j' dx &= \int_{-\infty}^{min(x_i,x_j)} e^{2x - x_i - x_j} dx - \int_{min(x_i,x_j)}^{max(x_i,x_j)} e^{min(x_i,x_j) - max(x_i,x_j)} dx \\ &\quad + \int_{max(x_i,x_j)}^{\infty} e^{x_i + x_j - 2x} dx \\ &= e^{-|x_i - x_j|} \big(1 - |x_i - x_j|\big). \end{aligned} \tag{2.31}$$

Inserting this into (2.29) yields

$$\|u(\cdot,t)\|^2_{H^1_{loc}(\mathbb{R}))} \leq \sum_{i,j=1}^{n} 2 m_i m_j e^{-|x_i - x_j|} dx < \infty \tag{2.32}$$

Our conclusive bound was based on the assumption that for any given $T > 0$ all m_i functions are bounded. An assumption that is valid because all m_i are C^1 functions and initially bounded. □

Remark 2.4. The ODEs in (2.10) are obviously valid as long as
$x_i(t) < x_{i+1}(t), \quad \forall i \in \{1, 2, .., n\}$. Peakons can however collide. What happens when peakons collide dependens on the value of b in the PDE (2.6). Restricting ourselves to $b \in [2,3]$ we can say a few things. As will be proved, if the multi-peakon only contains peakons or anti-peakons, collisions will not occur. But if the multi-peakon contains both anti-peakons and peakons, at least one peakon collision will occur if and only if the left-most peakon lies to the left of the rightmost antipeakon. What happens at and after a collision is an area of research, but for $b \in \{2,3\}$, much is known for simple peakon anti-peakon cases.

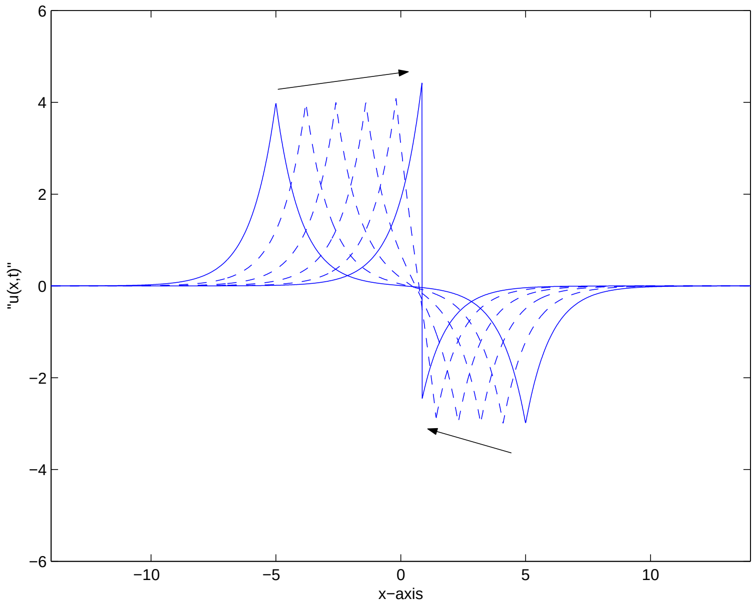

Figure 2.2: Matlab simulation of a peakon anti-peakon collision for the DP equation based on the ODEs (2.37) with inital values $(x_1(0) = -5, m_1(0) = 4), (x_2(0) = 5, m_2(0) = -3)$. The function $u(x,t)$ is shown at uniformly sampled times with dotted lines being intermediate samples, and complete lines being first and last time sample. As indicated by the arrows does the anti-peakon move leftward and the peakon rightward until they meet at a space-time point $(z, \tilde{t})$ where $u_x(z, \tilde{t}) = -\infty$. Continuation of u after collision for the DP equation is studied in chapter 4.

2.2 Conservation properties

By equation (2.10) we see that CH multi-peakons ($b = 2$) have to satisfy the system

$$\begin{aligned} \dot{x}_i &= \sum_{j=1}^{n} m_j e^{-|x_i - x_j|} \\ \dot{m}_i &= m_i \sum_{j=1}^{n} m_j \operatorname{sign}(x_i - x_j) e^{-|x_i - x_j|}. \end{aligned} \tag{2.33}$$

(2.33) is a Hamiltonian system

$$H = \frac{1}{2} \sum_{i,j=1}^{n} m_i m_j e^{-|x_i - x_j|}. \tag{2.34}$$

It means that (2.33) can be written as

$$\dot{x}_i = \frac{\partial H}{\partial m_i}, \quad \dot{m}_i = -\frac{\partial H}{\partial x_i}. \tag{2.35}$$

This gives the CH equation a lot of nice conservation properties. For example, for multi-peakons the $H^1(\mathbb{R})$ norm is conserved w.r.t. time. We see this by first showing that

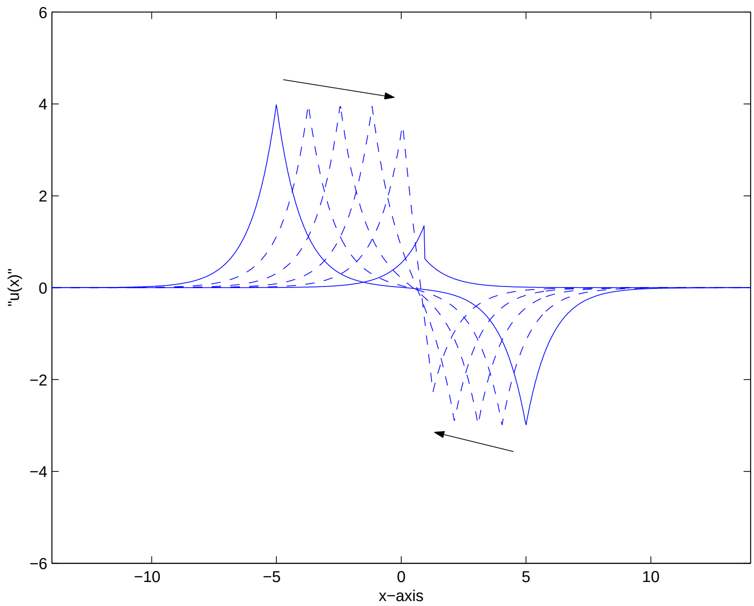

Figure 2.3: Matlab simulation of a peakon anti-peakon collision for the CH equation based on the ODEs (2.33) with inital values $(x_1(0) = -5, m_1(0) = 4), (x_2(0) = 5, m_2(0) = -3)$. The skew arrows indicate a dampening of u before collision which we did not see equally strong in the DP case, figure 2.2. This dampening makes u converge into one peakon/anti-peakon with momentum $m_1 + m_2$.

$\|u\|^2_{H^1(\mathbb{R})}$ is a linear function of H, and then showing that $\dot{H} = 0$.

$$
\begin{aligned}
u(x) \quad &= \textstyle\sum_{i=1}^n m_i e^{-|x-x_i|} &&\Longrightarrow \\
\|u\|^2_{H^1(\mathbb{R})} \quad &= \langle u - u_{xx}, u\rangle_{H^{-1}} &&= \textstyle\sum_{i=1}^n 2m_i u(x_i) \\
&= 2\textstyle\sum_{i,j=1}^n m_i m_j e^{-|x_i-x_j|} &&= 4H, \\
\dot{H} \quad &= \textstyle\sum_{i=1}^n \frac{\partial H}{\partial m_i}\dot{m}_i + \frac{\partial H}{\partial x_i}\dot{x}_i \\
&= \textstyle\sum_{i=1}^n \dot{x}_i \dot{m}_i - \dot{m}_i \dot{x}_i &&= 0 \\
\Longrightarrow \quad & \tfrac{d}{dt}\|u\|_{H^1(\mathbb{R})} &&= 0.
\end{aligned}
\tag{2.36}
$$

The DP multi-peakons have to satisfy the system

$$
\begin{aligned}
\dot{x}_i &= \sum_{j=1}^n m_j e^{-|x_i-x_j|}, \\
\dot{m}_i &= 2m_i \sum_{j=1}^n m_j \operatorname{sign}(x_i - x_j) e^{-|x_i-x_j|}.
\end{aligned}
\tag{2.37}
$$

The DP and the CH equation have many conservation properties in common. The spesifics proven here are done for the more general familiy of equations (2.6) with $b \in [2,3]$ or $b \in \mathbb{R}$.

Lemma 2.5. *For multi-peakon weak solutions of* (2.6) *the following properties are fulfilled*

(i)

$$\sum_{i=1}^n \dot{m}_i = 0, \quad \forall b \in \mathbb{R}.$$

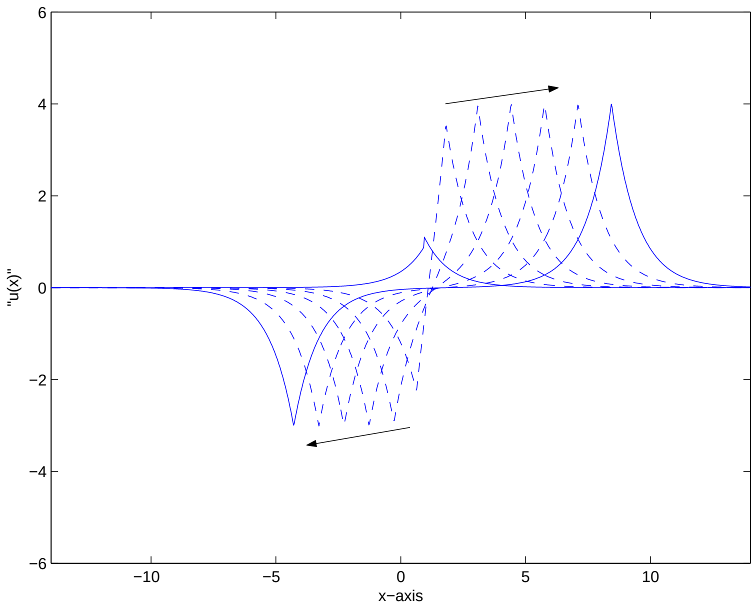

Figure 2.4: Continuation of u after collision for the CH equation example in figure 2.3. This continuation is not unique, but corresponds nicely with the conservation of H^1 norm mentioned in (2.36). It also gives the peakon and anti-peakon the soliton property of emerging from collisions unchanged. Another way to continue u after the collision is to "melt" the peakon and the anti-peakon into one peakon/anti-peakon with momentum $m_1 + m_2$.

(ii) *If $m_i(0) > 0$, then $m_i(t) \geq 0, \quad \forall t \in \mathbb{R}_+, \quad \forall b \in \mathbb{R}$. The analogue situation $m_i(0) < 0 \implies m_i(t) \leq 0$ is also true.*

(iii) *For a multi-peakon $u = \sum_{i=1}^n m_i e^{-|x-x_i|}$ consisting only of peakons/anti-peakons, all peakons/anti-peakons are uniformly bounded in absolute value by $|\sum_{i=1}^n m_i|$.*

(iv) *If $b \in [2,3]$ and the multi-peakon $u = \sum_{i=1}^n m_i e^{-|x-x_i|}$ is of such a nature that all its momenta are uniformly bounded in absolute value*

$$|m_i(t)| \leq M \qquad \forall t \in \mathbb{R}_+, \quad \forall i \in \{1,2,...,n\}, \tag{2.38}$$

then no collisions, either peakon/peakon or peakon/anti-peakon, will occur.

Proof.

(i)

$$\begin{aligned} \sum_{i=1}^n \dot{m}_i &= (b-1) \sum_{i,j=1}^n m_i m_j \operatorname{sign}(x_i - x_j) e^{-|x_i - x_j|} \\ &= 0 \end{aligned} \tag{2.39}$$

(ii) This follows directly from (2.10) since if for given time $m_i(t) = 0$, then $\dot{m}_i(t) = 0$.

(iii) Observe that by (i) and (ii)

$$|m_i(t)| \leq \Big|\sum_{j=1}^n m_j(t)\Big| = \Big|\sum_{j=1}^n m_j(0)\Big|$$

(iv) Given the function

$$f(t) = (\prod_{i=1}^{n} m_i) \prod_{i=1}^{n-1} (1 - e^{x_i - x_{i+1}})^{(b-1)}, \tag{2.40}$$

then $x_i(t) = x_{i+1}(t)$ if and only if $f(t) = 0$. So if we can prove that $\dot{f} = 0$, then $f(t) = f(0) \neq 0$ and we have almost completed the proof.

$$\begin{aligned}
\dot{f} &= \sum_{i=1}^{n} \frac{\dot{m}_i}{m_i} f(t) - (b-1) \sum_{i=1}^{n-1} \frac{(\dot{x}_i - \dot{x}_{i+1}) e^{x_i - x_{i+1}}}{1 - e^{x_i - x_{i+1}}} f(t) \\
&= \sum_{i=1}^{n} \frac{\dot{m}_i}{m_i} f(t) - (b-1) \sum_{i=1}^{n-1} \frac{(u(x_i) - u(x_{i+1})) e^{x_i - x_{i+1}}}{1 - e^{x_i - x_{i+1}}} f(t) \\
&= \sum_{i=1}^{n} \frac{\dot{m}_i}{m_i} f(t) \\
&\quad - (b-1) \sum_{i=1}^{n-1} \Big(\sum_{j=1}^{i} m_j e^{x_j - x_i} - \sum_{j=i+1}^{n} m_j e^{x_{i+1} - x_j} \Big) e^{x_i - x_{i+1}} f(t) \\
&= \Big(\sum_{i=1}^{n} \frac{\dot{m}_i}{m_i} - (b-1) \sum_{i=1}^{n-1} \big(\sum_{j=1}^{i} m_j e^{x_j - x_{i+1}} - \sum_{j=i+1}^{n} m_j e^{x_i - x_j} \big) \Big) f(t) \\
&= \Big(\sum_{i=1}^{n} \frac{\dot{m}_i}{m_i} - (b-1) \sum_{i=1}^{n} \big(\sum_{j=1}^{i-1} m_j e^{x_j - x_i} - \sum_{j=i+1}^{n} m_j e^{x_i - x_j} \big) \Big) f(t) \\
&= \Big(\sum_{i=1}^{n} \frac{\dot{m}_i}{m_i} - (b-1) \sum_{i=1}^{n} \big(\sum_{j=1}^{n} m_j \operatorname{sign}(x_i - x_j) e^{-|x_j - x_i|} \big) \Big) f(t) \\
&= \Big(\sum_{i=1}^{n} \frac{\dot{m}_i}{m_i} - \sum_{i=1}^{n} \frac{\dot{m}_i}{m_i} \Big) f(t) = 0
\end{aligned} \tag{2.41}$$

Here we have used that

$$u(x_i) - u(x_{i+1}) = \Big(\sum_{j=1}^{i} m_j e^{x_j - x_i} - \sum_{j=i+1}^{n} m_j e^{x_{i+1} - x_j} \Big) (1 - e^{x_i - x_{i+1}}).$$

Algebraically, equation (2.41) is valid even if momenta are not bounded. In particular, if a collision occurs in a multi-peakon between, say, the peakon (x_k, m_k) and the anti-peakon (x_{k+1}, m_{k+1}) at a given time $\tilde{t}$, then $\dot{f} = 0$ for $t < \tilde{t}$. Since $\lim_{t \to \tilde{t}^-} (x_k(t) - x_{k+1}(t)) = 0$ and

$$f(t) = (\prod_{i=1}^{n} m_i) \prod_{i=1}^{n-1} (1 - e^{x_i - x_{i+1}})^{(b-1)}, \tag{2.42}$$

is constant, its product term $\prod_{i=1}^{n} m_i$ must go to plus/minus infinity, meaning that some of its momenta must go to plus/minus infinity. Consequently, combining equation (2.41) with the assumption that all momenta are uniformly bounded in absolute value, proves that collisions will not occur.

□

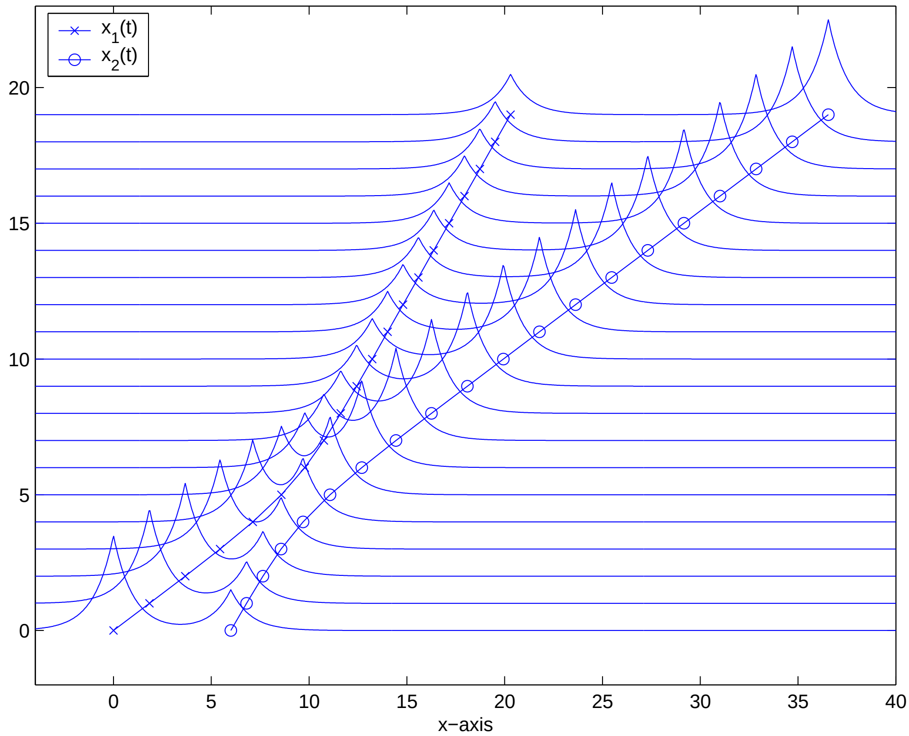

Figure 2.5: A two peakon DP multi-peakon simulation with initial conditions $(x_1(0) = 0, m_1(0) = 3.5), (x_2(0) = 6, m_2(0) = 1.5)$. The multi-peakon $u(x,t)$ is shown at evenly sampled times. Time is illustrated by translation of $u(x,t_k)$ along the y-axis; $u(x,t_k) = u(x,t_k) + k$, where t_k is the $(k+1)$th time sample. In the beginning, the strong peakon to the left is moving faster than the weaker peakon to the right. However, as claimed in Lemma 2.5, they do not collide. As the leftmost peakon moves towards the rightmost, momentum strength is transfered from the leftmost to the rightmost peakon. After some time the rightmost peakon has stronger momentum than the leftmost. Then, they move apart.

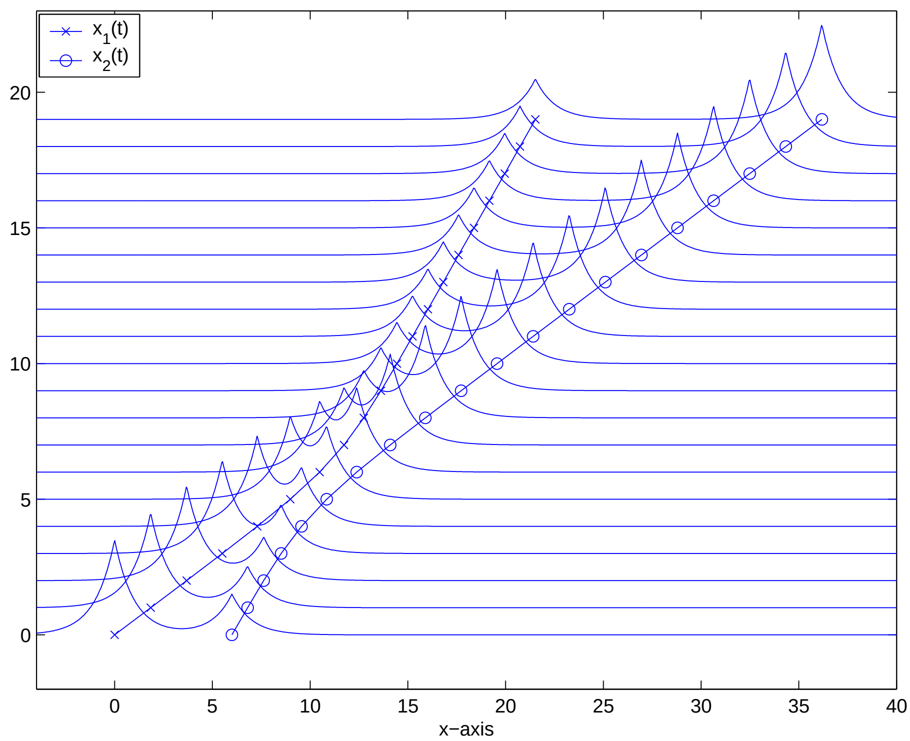

Figure 2.6: The analogue of the simulation in figure 2.5 for the CH multi-peakon, i.e. identical peakon inital conditions. This time the peakons come closer to each other before separation, and separation takes longer time than in the DP case. This is so because the $\dot{m}_i$ ODEs in (2.10) are smaller in absolute value in the CH case ($b = 2$) than in the DP case ($b = 3$).

Chapter 3

Entropy weak solution

According to definition 2.2 a weak solution u of the family of equations (2.6) has the property that $u(\cdot,t)$ belongs to H^1_{loc}. This restriction is natural to impose on u to make distributional sense of the $\frac{b-3}{2}u_x^2$ integrand term in (2.7). For the DP case ($b = 3$) the term disappears. It turns out that the H^1_{loc} restriction on weak DP solutions is stronger than necessary to achieve well-posedness. This chapter defines the DP entropy weak solution, a less restricted weak solution of the DP equation.

3.1 General weak solution

We consider the Dirichlet problem

$$\begin{aligned} (1-\partial_x^2)v(x) &= f(x), \quad x \in \mathbb{R}, \\ v(-\infty) = v(\infty) &= 0. \end{aligned} \tag{3.1}$$

Its Green's function, denoted $F(x,y)$ here (to avoid confusion since G is already taken), is defined as the solution of (3.1) with $f(x) = \delta(x-y)$. That is

$$F(x,y) = \frac{1}{2}e^{-|x-y|}. \tag{3.2}$$

Let $*$ be the convolution operator in Lebesgue sense, then

$$\begin{aligned} F*(1-\partial_x^2)v(x) &= \int_{\mathbb{R}} \frac{1}{2}e^{-|y|}(1-\partial_x^2)v(x-y)\,dy \\ &= \int_{-\infty}^{\infty} (1-\partial_x^2)(e^{-|x-z|}v(z)) \quad dz \\ &= \int_{-\infty}^{\infty} \left(e^{-|x-z|} - \partial_x\Big(\frac{-\operatorname{sign}(x-z)}{2}e^{-|x-z|}\Big)\right) v(z)\,dz \\ &= \int_{\mathbb{R}} \left(\Big(\frac{1}{2} - \frac{\operatorname{sign}(x-z)^2}{2}\chi_{\mathbb{R}\setminus\{x\}}(z)\Big)e^{-|x-z|} \right. \\ &\qquad \left. + \frac{(\operatorname{sign}(x^+-z) - \operatorname{sign}(x^--z))}{2(x^+-x^-)}e^{-|x-z|}\right) v(z)\,dz \\ &= \int_{\mathbb{R}} \Big(\delta(x-z) + \frac{\chi_{\{x\}}(z)}{2}\Big)v(z)\,dz = v(x) \end{aligned} \tag{3.3}$$

And likewise $(1-\partial_x^2)F*v(x) = v(x)$. Hence $(1-\partial_x^2)$ and $F*$ are formal inverse operators. This property will be used to reformulate what a weak solution of the PDE (2.6) is. We start by rewriting (2.6) as

$$
\begin{aligned}
0 &= u_t - u_{xxt} + \frac{b+1}{2}(u^2)_x + \frac{3-b}{2}(u_x^2)_x - \frac{1}{2}(u^2)_{xxx} \\
&= (u-u_{xx})_t + ((\frac{1}{2}u^2)_x - (\frac{1}{2}u^2)_{xxx}) + \frac{b}{2}(u^2)_x + \frac{3-b}{2}(u_x^2)_x \\
&= (1-\delta_x^2)(u_t + (\frac{1}{2}u^2)_x) + \frac{b}{2}(u^2)_x + \frac{3-b}{2}(u_x^2)_x
\end{aligned}
\tag{3.4}
$$

Applying the operator $F*$ on both sides of (3.4) gives us a conservation law

$$
\begin{aligned}
0 &= F * \Big((1-\partial_x^2)(u_t + (\frac{1}{2}u^2)_x) + \frac{b}{2}(u^2)_x + \frac{3-b}{2}(u_x^2)_x\Big) \\
&= (u_t + (\frac{1}{2}u^2)_x) + F * (\frac{b}{2}(u^2)_x + \frac{3-b}{2}(u_x^2)_x) \\
&= u_t + \partial_x \Big[\frac{1}{2}u^2 + F * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2)\Big].
\end{aligned}
\tag{3.5}
$$

Based on the conservation law (3.5) we reformulate definition 2.2 for $b \in \mathbb{R} \setminus \{3\}$.

Definition 3.1 (Reformulation of weak solution). $u \in L^\infty([0,T); H^1_{loc}(\mathbb{R}))$ is a weak solution of

$$
u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \qquad b \in \mathbb{R} \setminus \{3\},
$$

if it satisfies

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}} \phi D_t u + \phi D_x \Big[\frac{1}{2}u^2 + F * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2)\Big]\, dxdt = 0,& \\
\forall \phi \in C_c^\infty(\mathbb{R} \times [0,T)).&
\end{aligned}
\tag{3.6}
$$

For the DP case ($b=3$) the integrand $\frac{3-b}{2}u_x^2$ in (3.6) disappears. The remaining terms in the integral makes it natural to weaken the weak solution restriction from H^1 to L^2. Furthermore Coclite and Karlsen [6] showed, by using a conservation law, that if the initial function is in L^2, then its solution $u(\cdot,t)$ is in L^2 for all t where it is defined. With this in mind, the DP weak solution is redefined.

Definition 3.2 ((Redefined DP weak solution)). A function $u \in L^\infty([0,T); L^2(\mathbb{R}))$ is a weak solution of the Cauchy problem

$$
\begin{aligned}
u_t - u_{xxt} + 4uu_x &= 3u_xu_{xx} + uu_{xxx}, \\
u(x,0) &= u_0(x),
\end{aligned}
\tag{3.7}
$$

if it satisfies

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}} \phi D_t u + \phi D_x \Big[\frac{1}{2}u^2 + P^u\Big]\, dxdt = 0,& \\
\forall \phi \in C_c^\infty(\mathbb{R} \times [0,T)),& \\
\text{with} \quad P^u := \frac{3}{2} F * u^2.&
\end{aligned}
\tag{3.8}
$$

3.2 DP entropy weak solution

To achieve well-posedness Coclite and Karlsen extended the DP weak solution by requiring solutions to be of bounded variation and fulfil a Kruzkov-type entropy condition.

Definition 3.3 (Entropy weak solution). A function $u \in L^\infty([0,T);L^2(\mathbb{R}))$ is an entropy weak solution of the Cauchy problem (3.7) if

(i) u is a weak solution in the sense of Definition 3.2.

(ii) $u \in L^\infty([0,T);BV(\mathbb{R}))$.

(iii) For any convex C^2 entropy $\eta : \mathbb{R} \to \mathbb{R}$ with corresponding entropy flux $q : \mathbb{R} \to \mathbb{R}$ defined by $q'(u) = \eta'(u)u$, there holds

$$\int_0^T \int_{\mathbb{R}} \Big(D_t\eta(u) + D_x q(u) + \eta'(u) D_x P^u \Big) \phi \, dxdt \leq 0, \qquad \forall \phi \in C_c^\infty(\mathbb{R} \times [0,T)), \text{ such that } \phi \geq 0. \tag{3.9}$$

This can also be written

$$\int_0^T \int_{\mathbb{R}} \eta(u)\partial_t\phi + q(u)\partial_x\phi - \eta'(u) D_x P^u \, dxdt + \int_{\mathbb{R}} \phi(x,0)\eta(u_0(x))\, dx \geq 0. \tag{3.10}$$

In (3.10), we can replace the distributional derivative operator D with the normal derivative operator ∂ when operating on ϕ since it is a smooth function.

Remark 3.4. The bounded variation is imposed to ensure strong compactness for a sequence of solutions, i.e. existence.
Uniqueness of a weak solution would be true if we knew that the chain rule holds on our weak solutions. However, our weak solutions include discontinuous solutions for which the chain rule does not hold. Therefore the Kruzkov entropy condition is imposed to give stability, and thereby uniqueness. This will be further investigated in chapter 5.

Remark 3.5. By considering the case

$$\eta(u) = ((u-k)^2 + \delta)^{\frac{1}{2}}, \quad k \in \mathbb{R}, \quad \delta > 0, \tag{3.11}$$

and letting $\delta \to 0$, we can reduce the analysis to the case where

$$\eta_k(u) = |u - k|, \tag{3.12}$$

with corresponding

$$q_k(u) = \operatorname{sign}(u-k)\Big(\frac{u^2}{2} - \frac{k^2}{2}\Big). \tag{3.13}$$

It is shown by Holden and Risebro [9] (page 26 & 27) that if $\|u\|_{L^\infty(\mathbb{R}\times[0,T))} \leq M$, then for any convex function $\eta(u)$ there exists a sequence of functions

$$g_n = \beta + \sum_{k=1}^{n} \eta_k(u) + \alpha_k u \tag{3.14}$$

such that $g_n \to \eta$ in $L^\infty(-M,M)$. From this they conclude that for all $u \in L^\infty(\mathbb{R} \times [0,T))$ it suffices to verify the entropy condition (3.10) for the set of functions (3.12) and (3.13), which is called the Kruzkov entropies.

3.3 Shockpeakons

DP entropy weak solutions can be discontinuous. Examples of such are shockpeakons. A shockpeakon is a function of the form

$$u(x,t) = m_1(t)G(x - x_1(t)) + s_1(t)G'(x - x_1(t)), \tag{3.15}$$

where G and G' are defined as in chapter 2:

$$\begin{aligned} G(x - x(t)) &:= e^{-|x - x(t)|}, \\ G'(x - x(t)) &:= -\operatorname{sign}(x - x(t))G(x - x(t)). \end{aligned} \tag{3.16}$$

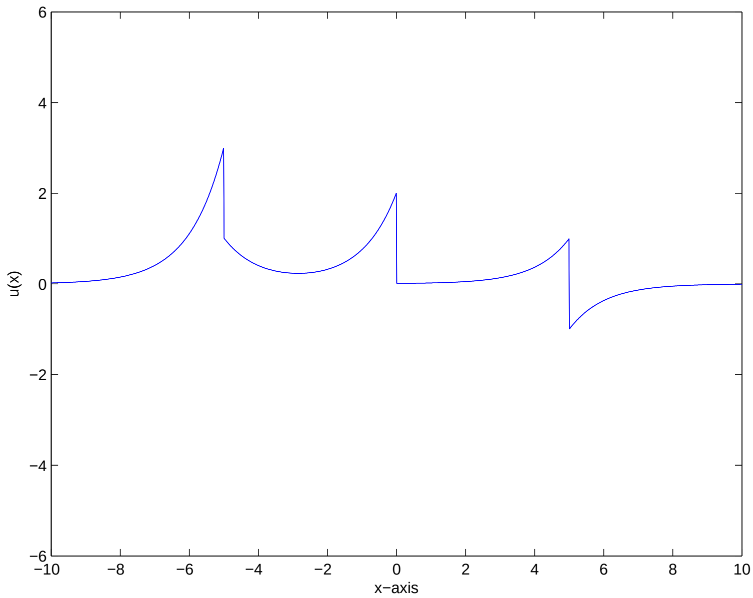

Figure 3.1: Illustration of a multi-shockpeakon ($\vec{x} = (-5, 0, 5)$, $\vec{m} = (2, 1, 0)$, $\vec{s} = (1, 1, 1)$)

Shockpeakons occur naturally as the limit function of a DP peakon anti-peakon collision. They consist of two parts, one peakon/anti-peakon part $m_1 G$ and one shock part $s_1 G'$. From (3.16) we see that the shockpeakon $u = m_1 G + s_1 G'$ has a jump discontinuity of $-2s_1$ at x_1 since

$$u(x,t) = \begin{cases} (m_1 + s_1)G(x - x_1), & x < x_1 \\ m_1, & x = x_1 \\ (m_1 - s_1)G(x - x_1), & x > x_1. \end{cases} \tag{3.17}$$

Multi-shockpeakons, which can be weak solutions of the DP equation, are defined the following way

Definition 3.6 (Multi-shockpeakon). A sum of shockpeakons

$$u = \sum_{i=1}^{n} m_i G_i + s_i G_i', \quad u \in L^\infty(\mathbb{R} \times [0, T)), \tag{3.18}$$

where $G_i := G(x - x_i)$ and $G_i' := G'(x - x_i)$, is called a multi-shockpeakon. Multi-shockpeakons are by convention sorted position-wisely;

$$-\infty < x_1(0) < x_2(0) < .... < x_{n-1}(0) < x_n(0) < \infty, \tag{3.19}$$

and all peakons, anti-peakons and shocks are bounded and differentiable on $[0,T)$

$$|m_i(0)|, |s_i(0)| < \infty, \quad \forall i \in \{1,2,...,n\}, \tag{3.20}$$

$$s_i(t), m_i(t), x_i(t) \in C^1([0,T)), \quad i \in \{1,2,...,n\}. \tag{3.21}$$

Theorem 3.7. *If* $u = \sum_{i=1}^n m_i G_i + s_i G_i'$ *then it satisfies the weak solution condition* (3.8) *if and only if*

$$\begin{aligned} \dot{x}_k &= u(x_k), \\ \dot{m}_k &= 2s_k u(x_k) - 2m_k\{u_x(x_k)\}, \\ \dot{s}_k &= -s_k\{u_x(x_k)\}, \end{aligned} \tag{3.22}$$

where

$$u(x_k) = \sum_{i=1}^n m_i(t)G(x_k - x_i) + s_i(t)G'(x_k - x_i), \tag{3.23}$$

and the curly brackets denote the nonsingular part

$$\{u_x(x_k)\} := \sum_{i=1}^n m_i(t)G'(x_k - x_i) + s_i(t)G(x_k - x_i). \tag{3.24}$$

Proof. We prove this, inspired by Lundmark's proof in [7], by calculating the distributional derivatives of u in (3.7). The proof is similar to the proof of theorem 2.3 involving some algebra on distributional derivatives.

Note from appendix A and the calculations in theorem 2.3 that

$$\begin{aligned} D_t(m_iG_i) &= \dot{m}_iG_i - \dot{x}_i m_i G_i', \\ \\ D_t(s_iG_i') &= \dot{s}_iG_i' + s_i(t)D_t(-\operatorname{sign}(x - x_i)G_i) \\ &= \dot{s}_iG_i' - s_i(t)\dot{x}_i(G_i - 2\delta_{x_i}), \\ \\ D_x(G_iG_j) &= G_jD_xG_i + G_iD_xG_j \\ &= G_jG_i' + G_iG_j'. \\ \\ D_x(G_i'G_j') &= G_j'D_xG_i' + G_i'D_xG_j' \\ &= G_j'(G_i - 2\delta_{x_i}) + G_i'(G_j' - 2\delta_{x_j}) \end{aligned} \tag{3.25}$$

Now, compute the distributional derivative terms D_tu and $D_x(u^2/2)$ of (3.7).

$$\begin{aligned} D_tu &= D_t(\sum_{i=1}^n m_iG_i + s_iG_i') \\ &= \sum_{i=1}^n \dot{m}_iG_i + m_iD_tG_i + \dot{s}_iG_i' + s_iD_tG_i' \\ &= \sum_{i=1}^n \dot{m}_iG_i - m_i\dot{x}_iG_i' + \dot{s}_iG_i' + s_i(2\dot{x}_i\delta_{x_i} - \dot{x}_iG_i) \\ &= \sum_{i=1}^n (\dot{m}_i - s_i\dot{x}_i)G_i + (\dot{s}_i - m_i\dot{x}_i)G_i' + 2s_i\dot{x}_i\delta_{x_i} \end{aligned} \tag{3.26}$$

Since u is a finite sum of normalized functions, it is a normalized function. So, by proposition A.5, D_x operated on u^2 satisfies the Leibniz rule $D_x u^2 = 2uD_x u$.

$$\begin{aligned}
D_x\frac{u^2}{2} &= \Big(\sum_{j=1}^{n} m_jG_j + s_jG_j'\Big)D_x\Big(\sum_{i=1}^{n} m_iG_i + s_iG_i'\Big) \\
&= \Big(\sum_{j=1}^{n} m_jG_j + s_jG_j'\Big)\Big(\sum_{i=1}^{n} m_iG_i' + s_i(G_i - 2\delta_{x_i})\Big) \qquad (3.27)\\
&= \sum_{i,j=1}^{n} m_im_jG_i'G_j + m_is_jG_i'G_j' + s_im_j(G_i - 2\delta_{x_i})G_j + s_is_jG_j'(G_i - 2\delta_{x_i}).
\end{aligned}$$

The next step is to compute $D_xP^u = D_x\frac{3}{2}F * u^2$, which is a bit more laborious. Recalling that $F(x,y) = \frac{1}{2}e^{-|x-y|} = \frac{1}{2}G(x-y)$ gives us

$$\begin{aligned}
D_xP^u(x,t) &= \frac{3}{4}\int_{\mathbb{R}} D_xG(x-y)u^2(y,t)dy \\
&= \frac{3}{4}\int_{\mathbb{R}} G'(x-y)u^2(y,t)dy \\
&= \frac{3}{4}\int_{\mathbb{R}} G'(x-y)\Big(\sum_{i,j=1}^{n} m_im_jG_iG_j + 2m_is_jG_iG_j' + s_is_jG_i'G_j'\Big)dy \qquad (3.28)\\
&= \frac{3}{4}\sum_{i,j=1}^{n} m_im_j\int_{\mathbb{R}} G'(x-y)G_iG_j\,dy + 2m_is_j\int_{\mathbb{R}} G'(x-y)G_iG_j'\,dy \\
&\quad + s_is_j\int_{\mathbb{R}} G'(x-y)G_i'G_j'\,dy.
\end{aligned}$$

To find $G' * (G_iG_j)$, $G' * (G_iG_j')$ and $G' * (G_i'G_j')$ we split $\mathbb{R}$ into at most three intervals. Presuming $x_i \leq x_j$ yields the splitting $I_1 = (-\infty, x_i)$, $I_2 = [x_i, , x_j)$ and $I_3 = [x_j, \infty)$.

$$\begin{aligned}
G' * (G_iG_j) &= \int_{I_1} G'(x-y)G(y-x_i)G(y-x_j)dy + \int_{I_2} G'(x-y)G(y-x_i)G(y-x_j)dy \\
&\quad + \int_{I_3} G'(x-y)G(y-x_i)G(y-x_j)dy \\
&= \int_{I_1} -\operatorname{sign}(x-y)e^{2y-(x_i+x_j)-|x-y|}dy + \int_{I_2} -\operatorname{sign}(x-y)e^{x_i-x_j-|x-y|}dy \\
&\quad + \int_{I_3} -\operatorname{sign}(x-y)e^{x_i+x_j-2y-|x-y|}dy \\
&= A + B + C.
\end{aligned} \qquad (3.29)$$

We look at the integrals separately.

$$A = \begin{cases} e^{x_i - x_j}\big(e^{x - x_i} - \frac{4}{3}e^{2x-2x_i}\big) & \text{if } x \le x_i, \\ -\frac{1}{3}e^{x_i - x_j}e^{x_i - x} & \text{if } x > x_i, \end{cases}$$

$$B = \begin{cases} e^{x_i - x_j}\big(e^{x - x_i} - e^{x - x_j}\big) & \text{if } x \le x_i, \\ e^{x_i - x_j}\big(e^{x_i - x} - e^{x - x_j}\big) & \text{if } x_i < x \le x_j, \\ e^{x_i - x_j}\big(e^{x_i - x} - e^{x_j - x}\big) & \text{if } x > x_j, \end{cases} \tag{3.30}$$

$$C = \begin{cases} \frac{1}{3}e^{x_i - x_j}e^{x - x_j} & \text{if } x \le x_j, \\ e^{x_i - x_j}\big(\frac{4}{3}e^{2x_j - 2x} - e^{x_j - x}\big) & \text{if } x > x_j. \end{cases}$$

Hence

$$G' * (G_i G_j) = \begin{cases} e^{x_i - x_j}\big(2e^{x - x_i} - \frac{4}{3}e^{2x-2x_i} - \frac{2}{3}e^{x - x_j}\big) & \text{if } x \le x_i \\ e^{x_i - x_j}\big(\frac{2}{3}e^{x_i - x} - \frac{2}{3}e^{x - x_j}\big) & \text{if } x_i \le x \le x_j \\ e^{x_i - x_j}\big(\frac{2}{3}e^{x_i - x} - 2e^{x_j - x} + \frac{4}{3}e^{2x_j - 2x}\big) & \text{if } x_i \le x \le x_j \end{cases} \tag{3.31}$$

which compactly looks like

$$G' * (G_i G_j) = \frac{2}{3}e^{x_i - x_j}(2G_i - 2G_j + G_i' + G_j') - \frac{2}{3}(G_i' G_j + G_i G_j'). \tag{3.32}$$

And if we remove the a priori condition $x_i \le x_j$, we get

$$G' * (G_i G_j) = \frac{2}{3}e^{-|x_i - x_j|}(2\,\text{sign}(x_i - x_j)(G_j - G_i) + G_i' + G_j') - \frac{2}{3}(G_i' G_j + G_i G_j'). \tag{3.33}$$

Computing $G' * (G_i G_j')$ and $G' * (G_i' G_j')$ in the same manner yields

$$\begin{aligned} G' * (G_i G_j) &= \frac{2}{3}e^{-|x_i - x_j|}\big(2\,\text{sign}(x_i - x_j)(G_j - G_i) + G_i' + G_j'\big) - \frac{2}{3}(G_i' G_j + G_i G_j'), \\ G' * (G_i G_j') &= \frac{2}{3}e^{-|x_i - x_j|}\big(\,\text{sign}(x_i - x_j)(G_j' - G_i') + 2G_i - G_j\big) - \frac{2}{3}(G_i G_j + G_i' G_j'), \\ G' * (G_i' G_j') &= \frac{2}{3}e^{-|x_i - x_j|}\big(-\text{sign}(x_i - x_j)(G_j - G_i) + G_i' + G_j'\big) - \frac{2}{3}(G_i' G_j + G_i G_j'). \end{aligned} \tag{3.34}$$

We insert these values into (3.28)

$$
\begin{aligned}
D_x P^u = \frac{1}{2}\sum_{i,j=1}^{n} & m_i m_j \Big(e^{-|x_i - x_j|} \big(2\,\mathrm{sign}(x_i - x_j)(G_j - G_i) + G_i' + G_j' \big) \\
& - (G_i' G_j + G_i G_j') \Big) \\
& + \sum_{i,j=1}^{n} m_i s_j \Big(e^{-|x_i - x_j|} \big(\mathrm{sign}(x_i - x_j)(G_j' - G_i') + 2G_i - G_j \big) \\
& - (G_i G_j + G_i' G_j') \Big) \\
& + \frac{1}{2}\sum_{i,j=1}^{n} s_i s_j \Big(e^{-|x_i - x_j|} \big(- \mathrm{sign}(x_i - x_j)(G_j - G_i) + G_i' + G_j' \big) \\
& - (G_i' G_j + G_i G_j') \Big) \\
= - & \sum_{i,j=1}^{n} \Big(m_i m_j G_i' G_j + m_i s_j (G_i G_j + G_i' G_j') + s_i s_j G_i' G_j \Big) \\
& + \sum_{i,j=1}^{n} e^{-|x_i - x_j|} \Big(\frac{1}{2} m_i m_j \Big(2\,\mathrm{sign}(x_i - x_j)(G_j - G_i) + G_i' + G_j' \Big) \\
& \quad + m_i s_j \Big(\mathrm{sign}(x_i - x_j)(G_j' - G_i') + 2G_i - G_j \Big) \\
& \quad + \frac{1}{2} s_i s_j \Big(- \mathrm{sign}(x_i - x_j)(G_j - G_i) + G_i' + G_j' \Big) \Big).
\end{aligned}
\tag{3.35}
$$

When adding (3.27) and (3.35) all $G_i G_j$, $G_i G_j'$ and $G_i' G_j'$ terms cancel out. We add on (3.26) and sort the terms by the linearly independent functions G_i, G_i' and δ_{x_i}. As remarked theorem 2.3, we use that $G_j \delta_{x_i} = G(x - x_j)\delta(x - x_i) = G(x_i - x_j)\delta_{x_i}$. The result is

$$
\begin{aligned}
0 &= D_t u + D_x \frac{1}{2} u^2 + D_x P^u \\
&= \sum_{i=1}^{n} \Big(\dot{m}_i - s_i \dot{x}_i + 2m_i \sum_{j=1}^{n} (s_j - m_j\,\mathrm{sign}(x_i - x_j)) e^{-|x_i - x_j|} \\
&\quad - s_i \sum_{j=1}^{n} (m_j - s_j\,\mathrm{sign}(x_i - x_j)) e^{-|x_i - x_j|} \Big) G_i \\
&\quad + \sum_{i=1}^{n} \Big(\dot{s}_i - m_i \dot{x}_i + m_i \sum_{j=1}^{n} (m_j - s_j\,\mathrm{sign}(x_i - x_j)) e^{-|x_i - x_j|} \\
&\quad + s_i \sum_{j=1}^{n} (s_j - m_j\,\mathrm{sign}(x_i - x_j)) e^{-|x_i - x_j|} \Big) G_i' \\
&\quad + \sum_{i=1}^{n} \Big(2s_i \dot{x}_i - 2si \sum_{j=1}^{n} (m_j - s_j\,\mathrm{sign}(x_i - x_j)) e^{-|x_i - x_j|} \Big) \delta_{x_i} \\
&= \sum_{i=1}^{n} \Big(\dot{m}_i - s_i \dot{x}_i + 2m_i \{u_x(x_i)\} - s_i u(x_i) \Big) G_i \\
&\quad + \sum_{i=1}^{n} \Big(\dot{s}_i - m_i \dot{x}_i + m_i u(x_i) + s_i \{u_x(x_i)\} \Big) G_i' + \sum_{i=1}^{n} \Big(2s_i \dot{x}_i - 2s_i u(x_i) \Big) \delta_{x_i}.
\end{aligned}
\tag{3.36}
$$

Since we assume the positions x_i are distinct, the set $(\cup_{i=1}^n G_i) \cup (\cup_{i=1}^n G_i') \cup (\cup_{i=1}^n \delta_{x_i})$ is linearly independent. Hence for (3.36) to be true, all factors of the G_i, G_i' and δ_{x_i} terms have to be zero. In other words/symbols:

$$\left.\begin{aligned} \dot{x}_i &= u(x_i) \\ \dot{m}_i &= 2s_i u(x_i) - 2m_i\{u_x(x_i)\} \\ \dot{s}_i &= -s_i\{u_x(x_i)\} \end{aligned}\right\} \; i = \{1, 2, .., n\}. \tag{3.37}$$

□

Remark 3.8. We see that in the shockless case, $s_i = 0 \, \forall i$, the ODEs (4.16) are equal to the multi-peakon ODEs obtained in chapter 2 for DP multi-peakons (2.37). Thus the conservation properties explored in chapter 2 are still valid for this case. Furthermore, observe that if $s_i(t_0) = 0$, then its derivative will be zero, so it will remain zero.

The second restriction on DP multi-shockpeakon solutions is imposed by the entropy condition. It forces all shocks to be positive.

Theorem 3.9. *Multi-shockpeakon weak solutions (definition 3.2) satisfy the entropy condition* (3.10) *if and only if* $s_i \geq 0$ *for all i. That is, all shocks must satisfy* $u(x_i^-) \geq u(x_i^+)$.

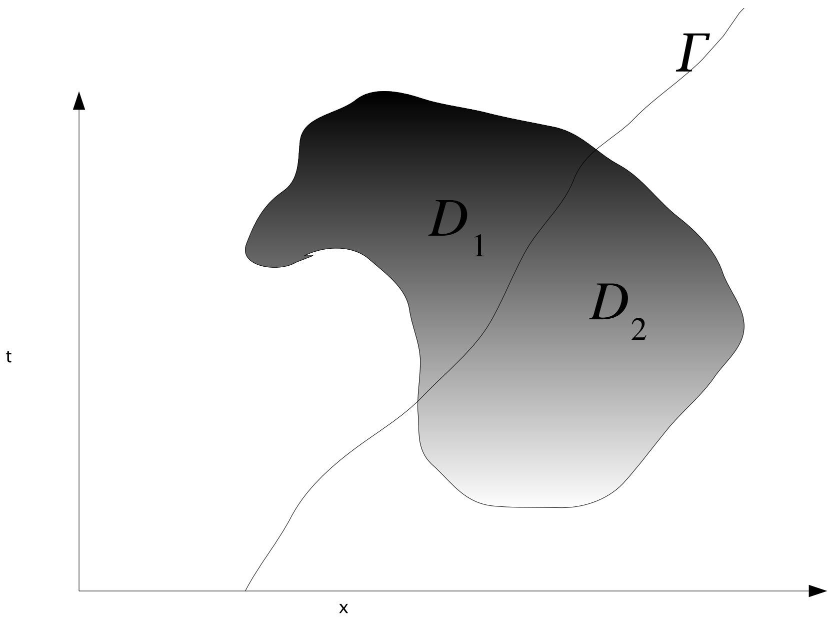

Figure 3.2: Isolated discontinuity curve Γ with an open divided neigbourhood around a point on the curve.

Proof. A multi-shockpeakon, $u = \sum_{i=1}^n m_i G_i + s_i G_i'$, is a piecewise strong solution. Its discontinuities with respect to x lies at the $x_i(t)$s. Up to shockpeakon collisions such discontinuities are isolated. Consider such an isolated discontinuity moving along a curve $\Gamma = (x_k(t), t)$, with left and right limits $u_l := u(x_k^-)$ and $u_r := u(x_k^+)$. Choose an open neighbourhood D around a point $(x_k(t_0), t_0)$ on the curve $(x_k(t), t)$ so small that u is smooth in D except on Γ, and a test function $\phi \geq 0$ whose support lies inside D. Then the entropy condition requires that

$$0 \le \iint_D \eta(u) D_t\phi + q(u) D_x\phi - \eta'(u)\phi D_x P^u \, dxdt. \tag{3.38}$$

Let D_1 and D_2 be the parts to the left and right of D, respectively. Then

$$\begin{aligned}
&\iint_{D_i} D_t(\eta(u)\phi) + D_x(q(u)\phi)\, dxdt - \iint_{D_i} \big(D_t\eta(u) + D_x q(u) + \eta'(u) D_x P^u\big)\phi \, dxdt \\
&= \int_{\partial D_i} \eta(u)\phi\nu_i^{(2)} + q(u)\phi\nu_i^{(1)} dS - \iint_{D_i} \big(D_t u + u D_x u + D_x P^u\big)\eta'(u)\phi\, dxdt.
\end{aligned} \tag{3.39}$$

Where

$$\nu_i(x,t) = (\nu_i^{(1)}, \nu_i^{(2)}) := (-1)^i \Big(\frac{-1}{\sqrt{1+\dot{x}_1^2}}, \frac{\dot{x}_1}{\sqrt{1+\dot{x}_1^2}}\Big)$$

is the outward pointing unit normal to ∂D_i. Since ϕ's support is contained inside D, we know that it is zero on the set $\partial D_i \cap \partial D$. Continuing on (3.39) we have

$$\begin{aligned}
\ldots &= \int_{\partial D_i \setminus \partial D} \eta(u)\phi\nu_i^{(2)} + q(u)\phi\nu_i^{(1)} dS - \iint_{D_i} \big(D_t u + u D_x u + D_x P^u\big)\eta'(u)\phi\, dxdt \\
&= (-1)^i \int_{\partial D_i \setminus \partial D} \eta(u)\phi \frac{\dot{x}_k}{\sqrt{1+\dot{x}_k^2}} - q(u)\phi\frac{1}{\sqrt{1+\dot{x}_k^2}} dS \\
&\quad - \iint_{D_i} \big(D_t u + u D_x u + D_x P^u\big)\eta'(u)\phi\, dxdt,
\end{aligned} \tag{3.40}$$

Noting that $D = D_1 \cup D_2$ in Lebesgue measure, we split the integral (3.38) into two pieces and use (3.40) to achieve

$$\begin{aligned}
0 &\le \iint_{D_1\cup D_2} \eta(u) D_t\phi + q(u) D_x\phi - \eta'(u)\phi D_x P^u \, dxdt \\
&= \iint_{D_1\cup D_2} D_t(\eta(u)\phi) + D_x(q(u)\phi)\, dxdt - \iint_{D_1\cup D_2} \big(D_t\eta(u) + D_x q(u) + \eta'(u) D_x P^u\big)\phi\, dxdt \\
&= \int_{\partial D_1 \setminus \partial D} \eta(u)\phi\frac{-\dot{x}_k}{\sqrt{1+\dot{x}_k^2}} + q(u)\phi\frac{1}{\sqrt{1+\dot{x}_k^2}}\, dS + \int_{\partial D_2\setminus\partial D} \eta(u)\phi\frac{\dot{x}_k}{\sqrt{1+\dot{x}_k^2}} - q(u)\phi\frac{1}{\sqrt{1+\dot{x}_k^2}} dS \\
&\quad + \iint_{D_1\cup D_2} \big(D_t u + u D_x u + D_x P^u\big)\eta'(u)\phi\, dxdt, \\
&= \int_\Gamma \Big((\eta(u_r) - \eta(u_l))\frac{\dot{x}_k}{\sqrt{1+\dot{x}_k^2}} - (q(u_r) - q(u_l))\frac{1}{\sqrt{1+\dot{x}_k^2}}\Big)\phi\, dS + 0.
\end{aligned} \tag{3.41}$$

This inequality is valid for all non-negative $\phi \in C_c^\infty(D)$. Thus

$$q(u_r) - q(u_l) \le \dot{x}_k(\eta(u_r) - \eta(u_l)) \qquad \text{on } \Gamma \tag{3.42}$$

Observe two things.
First, since x_k is a peakon position, we have from (3.22) that $\dot{x}_k = u(x_k)$. Furthermore, a DP weak solution multi-shockpeakon is a sum of normalized functions and thereby a normalized function itself, giving $\dot{x}_k = \frac{u_r+u_l}{2}$.

Second, by remark 3.5 we know that u satisfies the entropy condition (3.10) if it satisfies it for all Kruzkov entropies

$$\eta(u) := |u-k|, \quad q(u) := \text{sign}(u-k)\frac{u^2-k^2}{2}, \quad k \in \mathbb{R}. \tag{3.43}$$

Combining this with inequality (3.42) implies

$$\text{sign}(u_r-k)(u_r^2-k^2) - \text{sign}(u_l-k)(u_l^2-k^2) \leq (u_l+u_r)(|u_r-k|-|u_l-k|), \quad \text{on } \Gamma,\ \forall k \in \mathbb{R}. \tag{3.44}$$

Noticing that $\text{sign}(u-k)(u_l-k) = |u-k|$, we remove redundant terms in (3.44) and get the inequality

$$0 \leq (u_l-k)|u_r-k| - (u_r-k)|u_l-k|), \quad \text{on } \Gamma,\ \forall k \in \mathbb{R}. \tag{3.45}$$

By

$$(u_l-k)|u_r-k| - (u_r-k)|u_l-k|) = \begin{cases} -\Big((u_l-k)(u_r-k) - (u_r-k)(u_l-k)\Big) = 0, & k \geq max(u_l, u_r) \\ < 0, & u_r > k > u_l \\ > 0, & u_l > k > u_r \\ \Big((u_l-k)(u_r-k) - (u_r-k)(u_l-k)\Big) = 0 & k \leq min(u_l, u_r), \end{cases} \tag{3.46}$$

we conclude that multi-shockpeakons satisfy the entropy condition (3.10) if and only if $u_l \geq u_r$.

□

3.4 Conservation property

Although most conservation properties of DP-peakon solutions no longer are valid when looking at shockpeakon solutions, the sum of momenta is conserved.

$$\sum_{i=1}^{n} \dot{m}_i = 0 \tag{3.47}$$

This is proven algebraically by looking at the $\dot{m}_i$ ODE in (3.22).

$$\begin{aligned} \sum_{i=1}^{n} \dot{m}_i &= 2\sum_{j=1}^{n}\Big(s_j u(x_j) - m_j\{u_x(x_j)\}\Big) \\ &= 2\sum_{i,j=1}^{n}\Big(s_j(m_iG_i + s_iG_i') - m_j(s_iG_i - m_iG_i')\Big) \\ &= 0 \end{aligned} \tag{3.48}$$

Chapter 4

Various DP entropy solutions

One of the difficulties of describing multi-shockpeakon solutions explicitly is collisions. When a multi-peakon consists of many peakons and anti-peakons close to each other, it is, because of the forces involved, difficult to analyze what happens at a peakon anti-peakon collision and how to continue the solution beyond a collision. Therefore we begin our analysis on single- and dual-shockpeakons (multi-shockpeakons consisting of one or two components, respectively), and move on to various multi-shockpeakon scenarios.
We state the dual-shockpeakon ODEs (3.22) here for convenience ($x_1(0) < x_2(0)$)

$$\begin{aligned}
\dot{x}_1 &= m_1 + (m_2 + s_2)e^{x_1 - x_2} \\
\dot{x}_2 &= m_2 + (m_1 - s_1)e^{x_1 - x_2} \\
\dot{m}_1 &= 2(s_1 - m_1)(m_2 + s_2)e^{x_1 - x_2} \\
\dot{m}_2 &= 2(m_1 - s_1)(m_2 + s_2)e^{x_1 - x_2} \\
\dot{s}_1 &= -s_1^2 - s_1(m_2 + s_2)e^{x_1 - x_2} \\
\dot{s}_2 &= -s_2^2 + s_2(m_1 - s_1)e^{x_1 - x_2}
\end{aligned} \tag{4.1}$$

4.1 Shockpeakon peakon scenarios

Having already described single peakon anti-peakons as travelling waves in chapter 2 (2.1), we look at the behavior of a single shockpeakon. That is, the function

$$u(x,t) = m_1(t)G(x - x_1(t)) + s_1(t)G'(x - x_1(t)), \tag{4.2}$$

with initial conditions $x_1(0), m_1(0) \in \mathbb{R}$ and $s_1(0) \in \mathbb{R}_+$.

Solving the ODEs of (3.22), we get

$$\begin{aligned}
\dot{m}_1 = 2s_1 m_1 - 2m_1 s_1 = 0 &\implies m_1(t) = m_1(0), \\
\dot{x}_1(t) = u(x_1(t)) = m_1(t) = m_1(0) &\implies x_1(t) = x_1(0) + m_1(0)t, \\
\dot{s}_1(t) = -s_1^2(t) &\implies s_1(t) = \frac{s_1(0)}{1 + ts_1(0)}.
\end{aligned} \tag{4.3}$$

So shockpeakons moves at the constant speed $m_1(0)$, has the same constant momentum value (corresponding nicely to (3.47)) and a decreasing shock converging to zero;

$$u(x,t) = m_1(0)G(x - m_1(0)t) + \frac{s_1(0)}{1 + ts_1(0)}G'(x - m_1(0)t). \tag{4.4}$$

From (2.5)(iii) we know that in the pure peakon case collisions do not occur. For shock-peakons consisting exclusively of peakons and shocks, however, collisions can occur. Here, we will prove the existence of collisions for some scenarios and investigate how to continue solutions beyond collisions.

Theorem 4.1. *Given a dual-shockpeakon of the type $u = \sum_{i=1}^{2} m_i G_i + s_i G_i'$ where $m_1(0), m_2(0) > 0$ and $s_i(0) \leq m_i(0),\ i \in \{1,2\}$, then one shockpeakon collision can occur.*

Proof. To show existence of such a collision, we look at the $n = 2$ ODEs in (4.1) for the scenario

$$m_1, m_2 > 0,\ m_1 \geq s_1 \text{ and } s_2(0) = 0. \tag{4.5}$$

$$\begin{aligned}
\dot{x}_1 &= m_1 + (m_2)e^{x_1 - x_2} \\
\dot{x}_2 &= m_2 + (m_1 - s_1)e^{x_1 - x_2} \\
\dot{m}_1 &= 2m_2(s_1 - m_1)e^{x_1 - x_2} \\
\dot{m}_2 &= 2m_2(m_1 - s_1)e^{x_1 - x_2} \\
\dot{s}_1 &= -s_1^2 - s_1 m_2 e^{x_1 - x_2} \\
\dot{s}_2 &= 0
\end{aligned} \tag{4.6}$$

Now, setting $M = m_1(0) + m_2(0)$ we know that $\dot{M} = 0$. Given the scenario (4.5), deduce a few things

1. $m_2(t) \geq 0$.
Pf. Assume $m_2(\tilde{t}) = 0$. Then $\dot{m}_2 = 0$.

2.$m_1(t) \geq 0$.
Pf. We know that $s_1(t), m_2(t) \geq 0$. This implies that if $m_1(t) = 0$ then $\dot{m}_1 \geq 0$.

3.$m_1(0) \geq s_1(0) \implies m_1(t) \geq s_1(t)$.
Pf. Assume that at a given time $\tilde{t}$, $m_1(\tilde{t}) = s_1(\tilde{t})$. Then

$$\begin{aligned}
\dot{m}_1(\tilde{t}) - \dot{s}_1(\tilde{t}) &= s_1^2 + m_2(3s_1 - 2m_1)e^{x_1 - x_2} \\
&= s_1^2 + m_2 m_1 e^{x_1 - x_2} \\
&\geq 0.
\end{aligned} \tag{4.7}$$

4.$m_1(0) \geq s_1(0) \implies -\dot{m}_2 = \dot{m}_1 \leq 0$.
Pf. Using from 3 that $m_1(t) \geq s_1(t)$ and from 2 that $m_2(t) \geq 0$, this is clear from looking at (4.6). Hence $0 \leq m_1(t) \leq m_1(0)$ and $\leq m_2(0) \leq m_2(t)$.

$5. m_1(0) \geq s_1(0) \implies m_2(t) \leq m_2(0)e^{2m_1(0)t}$.
Pf. Prove this by using 3 and the inequality

$$\dot{m}_2(t) = 2m_2(m_1 - s_1)e^{x_1 - x_2} \leq 2m_1(0)m_2(t). \tag{4.8}$$

Similarly we have that

$$\begin{aligned} \dot{m}_1(t) \geq -2m_1(t)m_2(t) \geq -2Mm_1(t), &\implies m_1(t) \geq m_1(0)e^{-2Mt}, \\ \dot{s}_1(t) \geq -s_1(t)M, &\implies s_1(t) \geq s_1(0)e^{-Mt}. \end{aligned} \tag{4.9}$$

6. If $m_1(0) - s_1(0) - m_2(0) \leq 0$ then $m_1(t) - s_1(t) - m_2(t) \leq 0$.
Pf. Computing that $\dot{m}_1 - \dot{m}_2 < 0$, we see this straight away.

If $m_1(0) \geq s_1(0)$ and 6 above is satisfied, we see that

$$\begin{aligned} \dot{x}_2 - \dot{x}_1 &= m_2 - m_1 + (m_1 - s_1 - m_2)e^{x_1 - x_2} \\ &\leq m_2(0)e^{2m_1(0)t} - m_1(0)e^{-2Mt} \end{aligned} \tag{4.10}$$

Choosing the intial function $\vec{x}(0) = (0, 0.01)$, $\vec{m}(0) = (2, 1)$ and $\vec{s}(0) = (1, 0)$, it complies with scenario (4.5), which means conditions 1-6 above are satisfied. Hence

$$\dot{x}_2(t) - \dot{x}_1(t) \leq e^{0.2} - 2e^{-0.3} \leq -0.2, \quad t \in [0, 0.05]. \tag{4.11}$$

A collision must occur because

$$\begin{aligned} x_2(0.05) - x_1(0.05) &= x_2(0) - x_1(0) + \int_0^{0.05} \dot{x}_2(\tau) - \dot{x}_1(\tau) d\tau \\ &\leq 0.01 - 0.2 \int_0^{0.05} d\tau = 0. \end{aligned} \tag{4.12}$$

□

Having shown the existence of certain dual-shockpeakon collisions we move on to multi-shockpeakon scenarios. Our wish is to find multi-shockpeakons behaving nicely even if collisions occur. In particular, we would like the components $\{x_i\}$, $\{m_i\}$ and $\{s_i\}$ not to jump in value at collisions.

Theorem 4.2. *Given a multi-shockpeakon of the type* $u = \sum_{i=1}^{n} m_i G_i + s_i G_i'$ *where* $m_i(0) > 0$ *and* $s_i(0) \leq m_i(0)$ $\forall i \in \{1, 2, .., n\}$, *then shockpeakon collisions can occur. At collisions, which might be between more than two shockpeakons, the colliding shockpeakons* $\{(x_i, m_i, s_i)\}_{i=i_1}^{i_2}$ *"melt" into a single shockpeakon with*

$$\begin{aligned} \hat{x} &= x_{i_1} = x_{i_1+1} = ... = x_{i_2}, \\ \hat{m} &= \sum_{i=i_1}^{i_2} m_i \quad \textit{and} \quad \hat{s} = \sum_{i=i_1}^{i_2} s_i. \end{aligned} \tag{4.13}$$

Proof. Existence of collisions for $n > 2$ follows easily from proving the collision behaviour (4.13), so we prove this first.

We begin by registering a semi-conservational property; if $m_i(0) \geq s_i(0),\ \forall i$ then $m_i(t) \geq s_i(t), \forall i$.

Assume $m_i(0) \geq s_i(0)$, and that there is a time $\tilde{t}$ such that $m_j(\tilde{t}) = s_j(\tilde{t})$ for one or more $j \in \{1, 2, ..., n\}$. Using the ODEs in (3.22) we see that

$$\begin{aligned}
\dot{m}_j - \dot{s}_j &= 2s_j u(x_j) - 2m_j\{u_x(x_j)\} + s_j\{u_x(x_j)\} \\
&= s_j(2u(x_j) - \{u_x(x_j)\}) \\
&= s_j\Big(\sum_{i=1}^{n} 2m_i G(x_j - x_i) + 2s_i G'(x_j - x_i) - s_i G(x_j - x_i) - m_i G'(x_j - x_i)\Big) \\
&\geq s_j\Big(\sum_{i=1}^{n} m_i G(x_j - x_i) + 2s_i G'(x_j - x_i) - m_i G'(x_j - x_i)\Big) \\
&= s_j\Big(2\sum_{i=1}^{j-1} \big((m_i - s_i)G(x_j - x_i)\big) + m_j + \sum_{i=j+1}^{n} 2s_i G(x_j - x_i)\Big) \\
&\geq 0.
\end{aligned} \tag{4.14}$$

So at the time $\tilde{t}$ we have $\dot{m}_j(\tilde{t}) - \dot{s}_j(\tilde{t}) \geq 0$, which implies that s_j always will be less or equal to m_j.

Writing $M = \sum_{i=1}^{n} m_i$ we see from the result above that $S = \sum_{i=1}^{n} s_i \leq M$. Furthermore,

$$\left.\begin{aligned} 0 \leq u(x,t) &\leq M + S \leq 2M, \\ |\{u_x(x,t)\}| &\leq M + S \leq 2M, \end{aligned}\right\} \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{4.15}$$

We use this result to bound the ODEs in (3.22):

$$\left.\begin{aligned} |\dot{x}_i| &= |u(x_i)| \leq 2M \\ |\dot{m}_i| &= |2s_i u(x_i) - 2m_i\{u_x(x_i)\}| \leq 8M^2 \\ |\dot{s}_i| &= |-s_i\{u_x(x_i)\}| \leq 2M^2 \end{aligned}\right\} \; i = \{1, 2, .., n\} \tag{4.16}$$

This means the x_i, m_i and s_i components are Lipschitz contiuous. So if a collision occurs between shockpeakons $\{(x_i, m_i, s_i)\}_{i=i_1}^{i_2}$ at time equals $\tilde{t}$, then none of the involved components will jump in value. Thus the continuation of the solution beyond a collision consists of the shockpeakons not involved in the collision, and the shockpeakon created by the collision, which is described described by the components

$$\begin{aligned}
\hat{x}(\tilde{t}) &= \lim_{t \to \tilde{t}^-} x_{i_1}(t) = \lim_{t \to \tilde{t}^-} x_{i_1+1}(t) = ... = \lim_{t \to \tilde{t}^-} x_{i_2}(t), \\
\hat{m}(\tilde{t}) &= \lim_{t \to \tilde{t}^-} \sum_{i=i_1}^{i_2} m_i(t), \\
\hat{s}(\tilde{t}) &= \lim_{t \to \tilde{t}^-} \sum_{i=i_1}^{i_2} s_i(t).
\end{aligned} \tag{4.17}$$

We conclude that given

$$u(x,t) = \sum_{i=1}^{n} m_i(t) G(x - x_i(t)) + s_i(t) G'(x - x_i(t)), \quad 0 \leq t < \tilde{t}, \tag{4.18}$$

the continuation beyond a collision described in (4.17) becomes

$$\begin{aligned}
u(x,t) = &\sum_{i=1}^{i_1-1} \Big(m_i(t) G(x - x_i(t)) + s_i(t) G'(x - x_i(t)) \Big) \\
&+ \hat{m}(t) G(x - \hat{x}(t)) + \hat{s}(t) G'(x - \hat{x}(t)) \\
&+ \sum_{i=i_2+1}^{n} m_i(t) G(x - x_i(t)) + s_i(t) G'(x - x_i(t)), \\
\text{for} \quad &\tilde{t} \le t < \text{next collision time.}
\end{aligned} \tag{4.19}$$

To prove the existence of collisions in the $n > 2$ shockpeakon setting, choose the following shockpeakons shock and momentum values

$$\begin{aligned}
&m_i(0) = m > 0, \quad \forall i \in \{1,2,...,n\}, \\
&s_1(0) = s_n(0) = 0, \\
&s_i(0) = m, \quad i \in \{2,3,..,n-1\}, \\
\implies \; &M = \sum_{i=1}^{n} m_i(0) = nm(0), \;\; S = \sum_{i=1}^{n} s_i(0) = (n-2)m.
\end{aligned} \tag{4.20}$$

For $t < \frac{m}{32M^2}$ we see from (4.16) that

$$\begin{aligned}
&m_1(t) \ge m - \tfrac{8M^2 m}{32M^2} = \tfrac{3m}{4} \\
&m_n(t) \le m + \tfrac{8M^2 m}{32M^2} = \tfrac{5m}{4} \\
&\left.\begin{aligned} s_i(t) &\ge m - \tfrac{2M^2 m}{32M^2} &&= \tfrac{15m}{16} \\ s_i(t) &\le m_i(t) &&\le \tfrac{5m}{4} \end{aligned}\right\} i \in \{2,3,...,n-1\}
\end{aligned} \tag{4.21}$$

Restrict the shockpeakon positions in the following manner

$$0 = x_1(0) < x_2(0) < ... < x_{n-1}(0) < x_n(0) < ln(2). \tag{4.22}$$

Assume $t < \frac{m}{32M^2}$, then

$$\begin{aligned}
\dot{x}_n(t) - \dot{x}_1(t) &= u(x_n) - u(x_1) \\
&= (m_n - m_1)(1 - G(x_1 - x_n)) \\
&+ \sum_{i=2}^{n-1} \Big((m_i(t) - s_i(t)) G(x_i - x_n) - (m_i(t) + s_i(t)) G(x_1 - x_i) \Big) \\
&\le \frac{m}{2}(1 - G(x_1 - x_n)) + \sum_{i=2}^{n-1} \Big(\frac{5m}{16} - \frac{30m}{16} G(x_1 - x_n) \Big) \\
&\le \frac{m}{4} + (n-2)(\frac{5m}{16} - \frac{15m}{16}) \\
&\le -\frac{3m}{8}.
\end{aligned} \tag{4.23}$$

This means that if (4.22) is satisfied, then the distance between shockpeakon number 1 and n will decrease while $t < \frac{m}{32M^2}$.

Choosing $x_n(0) < \frac{m^2}{128M^2} = \frac{1}{128n^2} < ln(2)$ we get

$$\begin{aligned} x_n(\frac{m}{32M^2}) - x_1(\frac{m}{32M^2}) &= x_n(0) - x_1(0) + \int_0^{\frac{m}{32M^2}} \dot{x}_n - \dot{x}_1 d\tau \\ &\leq \frac{m^2}{128M^2} - \frac{m}{32M^2}\frac{3m}{8} < 0 \end{aligned} \tag{4.24}$$

Which means that at least one shockpeakon collision occurs before $t = \frac{m}{32M^2}$. □

Remark 4.3. To show the existence of collisions in our multi-shockpeakon subspace, we have looked at examples where the peakons initialy are perversely close to each other. As shown in figure 4.1, initial shockpeakon closeness is not that important for collisions to occur.

Remark 4.4. As will be shown in the next section, in spaces with peakons and anti-peakons the momentum components m_i can jump in value at collisions. This makes functions in that space hard to analyse because it is hard, if at all possible, to simulate such solutions numerically. The space of shockpeakons where we have only positive shocks and momenta, $s_i, m_i > 0$, and allow the shocks to be bigger than momenta, $s_i > m_i$, can probably experience the same problem. What makes this space hard to study is that the peakon part of a shockpeakon can be turned into an anti-peakon.

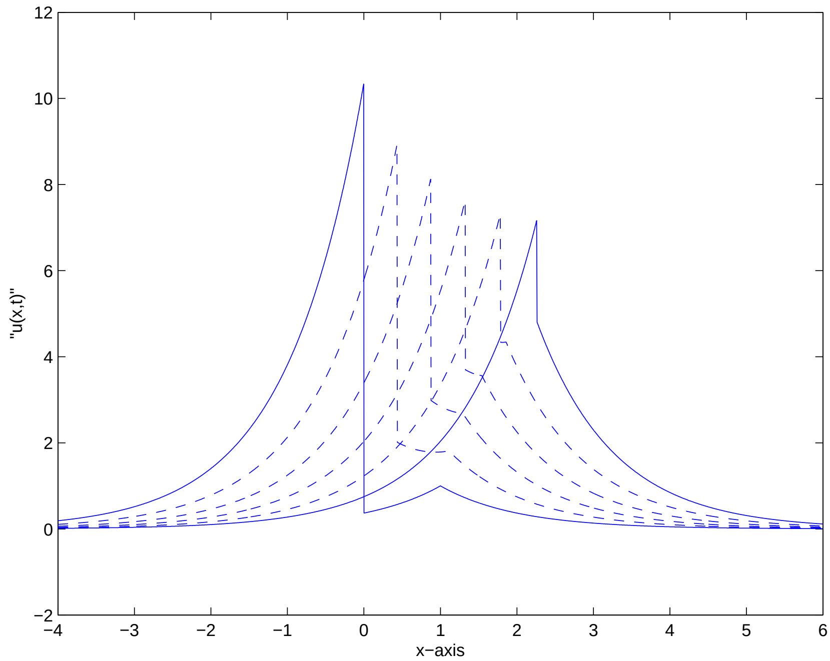

Figure 4.1: Shockpeakons with initial values $\vec{x} = (0, 1)$, $\vec{m} = (5, 1)$ and $\vec{s} = (5, 0)$ colliding.

4.2 Peakon anti-peakon scenarios

Lundmark and Szmigielski have done a lot of work on explicit multi-shockpeakon entropy solutions before and after shockpeakon collisions. In this section I will describe some of

their results from [7] and [8]. The theory is somewhat peripheral to what we are working on in later chapters, but it is included because it probably holds information on where to go next.
Looking at pure peakon multi-peakons, that is the set of functions

$$\Big\{u = \sum_{i=1}^{n} m_i G_i \Big| m_i \geq 0 \quad \forall i\Big\}, \tag{4.25}$$

explicit solutions were found using inverse scattering theory. For $n = 2$ the solution reads

$$\begin{aligned} x_1(t) &= \log \frac{\frac{(\lambda_1-\lambda_2)^2}{\lambda_1+\lambda_2} b_1 b_2}{\lambda_1 b_1 + \lambda_2 b_2}, \\ x_2(t) &= \log(b_1 + b_2), \\ m_1(t) &= \frac{(\lambda_1 b_1 + \lambda_2 b_2)^2}{\lambda_1\lambda_2\Big(\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2\Big)}, \\ m_2(t) &= \frac{(b_1 + b_2)^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} b_1 b_2}. \end{aligned} \tag{4.26}$$

Here we presume initial values $x_1(0), x_2(0), m_1(0)$ and $m_2(0)$ are given. λ_1 and λ_2 are distinct eigenvalues (constants) of a spectral problem, and determined through the relation

$$1 - (m_1 + m_2)z + m_1 m_2 \Big(1 - \frac{e^{x_1}}{e^{x_2}}\Big)^2 z^2 = \big(1 - \frac{z}{\lambda_1}\big)\big(1 - \frac{z}{\lambda_2}\big) \tag{4.27}$$

The residues $b_i(t) = b_i(0)e^{t/\lambda_i}$ are functions whose initial values are determined through the relation

$$\begin{aligned} b_1 + b_2 &= e^{x_2}, \\ \frac{b_1}{\lambda_1} + \frac{b_2}{\lambda_2} &= m_1 e^{x_1} + m_2 e^{x_2}. \end{aligned} \tag{4.28}$$

For $n \geq 2$ similar equations give the solution in the pure peakon case. Quoting Lundmark : "The general solution for $n \geq 2$ is given in terms of eigenvalues and Weyl function residues $\{\lambda_i, b_i\}_{i=1}^{n}$ of the "discrete cubic string", a third order nonselfadjoint spectral problem related to the Lax pair of the DP equation [8]. The eigenvalues λ_k are positive and distinct, the residues b_k are positive, the peakons behave like free particles with distinct speeds $\lambda_n^{-1} < \ldots < \lambda_1^{-1}$ as $t \to \infty$, and no collisions will occur."

Looking at the peakon anti-peakon case $n = 2$ we see from (4.27) that there are as many positive/negative eigenvalues as there are positive/negative momenta. In general, distinct eigenvalues in the sense

$$\lambda_i \neq \pm\lambda_j, \quad \text{when } i \neq j, \tag{4.29}$$

is necessary for these solutions to be valid. For example, in the case $n = 2$ we see that if $\lambda_1 = -\lambda_2$ the solution does not make sense (we will look at this special case in theorem 4.5). For general multi-peakon solutions $n > 2$, not much is known. One suspects that the solution dynamics represented by $n = 2$ might be extendable to the general case. That is, solutions are locally valid if eigenvalues are real and distinct, and there are as many positive/negative eigenvalues as there are positive/negative momenta.

Here we leave the general case and give a summary of Lundmark's results in peakon anti-peakon dynamics for $n = 2$.

Theorem 4.5 (Genesis of shockpeakons). *The solution of the $n = 2$ DP peakon ODEs (3.22) in the symmetric peakon anti-peakon case $m_1 + m_2 = 0$ with $-x_1(0) = x_2(0) > 0$ is given by*

$$\begin{aligned} -x_1(t) &= x_2(t) = x_2(0) - \frac{t}{\lambda}, \\ m_1(t) &= -m_2(t) = \frac{1}{\lambda(1 - e^{2x_2(t)})}, \end{aligned} \tag{4.30}$$

where $\lambda = \Big(m_1(0)(1 - e^{-2x_2(0)}\Big)^{-1}$.

(i) If $m_1(0) < 0 < m_2(0)$, then $\lambda < 0$ and the solution (4.30) is valid for $t > t_{min}$, where $t_{min} := \lambda x_2(0) < 0$. In particular, $u = \sum_{k=1}^{2} m_k e^{-|x-x_k|}$ provides a solution of the initial value problem which is valid for all $t \geq 0$.

(ii) If $m_1(0) > 0 > m_2(0)$, then $\lambda > 0$ and a collision occurs at $x = 0$ for $t = t_0 = \lambda x_2(0) > 0$. The function $u = \sum_{k=1}^{2} m_k e^{-|x-x_k|}$ only satisfies the DP entropy solution criterions 3.3 for $t < t_0$. The unique entropy solution continuation of $u(x,t)$ is given by the stationary decaying shockpeakon

$$u(x,t) = \frac{-\operatorname{sign}(x)e^{-|x|}}{\lambda + (t - t_0)}, \quad \text{for } t \geq t_0. \tag{4.31}$$

Proof. By the ODEs in (4.1) we have

$$\dot{x}_1 = m_1(1 - e^{-2x_2(t)}), \qquad m_1 = \frac{-\dot{x}_2}{(1 - e^{-2x_2(t)})}, \tag{4.32}$$

$$\begin{aligned} \dot{m}_1 = 2(m_1)^2 e^{-2x_2(t)} &= \frac{m_1(-2\dot{x}_2 e^{-2x_2(t)})}{(1 - e^{-2x_2(t)})} \\ \implies m_1(t) &= \frac{m_1(0)(1 - e^{-2x_2(0)})}{(1 - e^{-2x_2(t)})}. \end{aligned} \tag{4.33}$$

Thus

$$\dot{x}_1 = m_1(0)(1 - e^{-2x_2(0)}) = \lambda \qquad \implies x_1(t) = x_1(0) + \lambda t. \tag{4.34}$$

This proves equation (4.30). Moving on, observe that

$$u(x_1(t),t) = m_1(t)(1 - e^{-2x_2(t)}) = -m_2(t)(1 - e^{-2x_2(t)}) = -u(x_2(t),t), \quad \forall t \in [0, t_0^-). \tag{4.35}$$

Hence unlike the Camassa-Holm case where $u \to 0$ uniformly at a collision, the peakon and the anti-peakon do not cancel completely out at a collision. Instead $u(x,t)$ converges as $t \to t_0^-$ to the discontinuous function

$$u(x,t_0) = -\frac{1}{\lambda}\operatorname{sign}(x)e^{-|x|} = \frac{1}{\lambda}G'(x) \tag{4.36}$$

with the convergence uniform on the interval not containing $x = 0$. A shock is formed at $x = 0$ with shock strength $s(t_0) = \lambda$. The unique entropy solution continuation of such a shock, described in (4.4), is the decreasing function

$$u(x,t) = \frac{1}{\lambda + (t - t_0)} G'(x). \tag{4.37}$$

□

Interestingly, the b value in the family of equation (2.6) is determining collision behaviour. Considering the problem in theorem 4.5 for general b values, using the ODEs in (2.10) we get

$$\begin{aligned} \dot{x}_1 = m_1(1 - e^{-2x_2(t)}) \geq 0, \quad m_1 &= \frac{-\dot{x}_2}{(1 - e^{-2x_2(t)})} \\ \dot{m}_1 = (b-1)(m_1)^2 e^{-2x_2(t)} &= \frac{m_1 \frac{(b-1)}{2} (-2\dot{x}_2 e^{-2x_2(t)})}{(1 - e^{-2x_2(t)})} \\ \implies m_1(t) &= \frac{m_1(0)(1 - e^{-2x_2(0)})^{(b-1)/2}}{(1 - e^{-2x_2(t)})^{(b-1)/2}}. \end{aligned} \tag{4.38}$$

By the same reasoning as in the theorem we have

$$u(x_1(t),t) = m_1(0)(1 - e^{-2x_2(0)})^{(b-1)/2}(1 - e^{-2x_2(t)})^{(3-b)/2} = -u(x_2(t),t) \tag{4.39}$$

for all times before collision. Thus if $b < 3$, then $u \to 0$ at collision time (for example in the Camassa-Holm case $b = 2$), and if $b > 3$, then $u \to \infty$ at collision time. Only for the DP equation, $b = 3$, do shocks form at collisions.

In the general $n = 2$ peakon anti-peakon scenarios we also see that a collision forms a shockpeakon.

Theorem 4.6. *The solution of the $n = 2$ DP peakon ODEs* (4.1) *with $x_1(0) < x_2(0)$ and $m_1 + m_2 \neq 0$ is given by* (4.26)*, where λ_k and $b_k(0)$ are determined from the initial conditions.*

- *If $m_1(0)$ and $m_2(0)$ have the same sign, then $u = \sum_{k=1}^{2} m_k e^{-|x-x_k|}$ together with* (4.26) *defines a global solution of the DP equation. In particular, as a solution of the initial value problem it is valid for all $t \geq 0$.*

- *If $m_1(0) < 0 < m_2(0)$, then* (4.26) *gives a valid solution of the DP equation for $t > t_{min}$, where*

$$t_{min} = \frac{-1}{\lambda_1^{-1} - \lambda_2^{-1}} \log \left(\frac{1-\kappa}{\kappa(1+\kappa)} \frac{b_1(0)}{b_2(0)} \right) < 0 \qquad (\kappa = \sqrt{-\lambda_2/\lambda_2}). \tag{4.40}$$

  In particular, as a solution of the initial value problem it is valid for all $t \geq 0$.

- *If $m_1(0) > 0 > m_2(0)$, then* (4.26) *gives a valid solution of the DP equation for $t < t_0$, where the time of collision t_0 is*

$$t_0 = \frac{1}{\lambda_1^{-1} - \lambda_2^{-1}} \log \left(\frac{\kappa(\kappa-1)}{1+\kappa} \frac{b_2(0)}{b_1(0)} \right) > 0 \qquad (\kappa = \sqrt{-\lambda_2/\lambda_2}). \tag{4.41}$$

The continuation of $u(x,t)$ into the unique entropy solution of the initial value problem is for $t \geq t_0$ given by the moving shockpeakon

$$u(x,t) = \Big(\tilde{m}_1 - \text{sign}(x - \tilde{x}_1(t))\tilde{s}_1(t)\Big)e^{-|x-\tilde{x}_1(t)|}, \tag{4.42}$$

where $\tilde{m} = m_1 + m_2 = \lambda_1^{-1} + \lambda_2^{-1}$ (constant), $\tilde{x}_1(t) = (t - t_0)\tilde{m}_1 + \tilde{x}_1(t_0)$ with $\tilde{x}_1(t_0) = x_1(t_0) = x_2(t_0)$ as the point of collision, and $\tilde{s}_1(t) = (t - t_0 + \tilde{s}_1(t_0)^{-1})^{-1}$ with $\tilde{s}_1(t_0) = \sqrt{-\lambda_1^{-1}\lambda_2^{-1}} > 0$.

Proof. See Lundmark's proof of theorem 3.5 in [7]. □

Knowing the solution to peakon anti-peakon scenarios, we would like to extend this knowledge to multi-peakon scenarios. One thing makes these peakon anti-peakon scenarios more difficult than the ones investigated in theorem 4.2; apart from $\sum_i \dot{m}_i = 0$ they do not possess any conservational properties on momenta. This can be seen from theorem 4.5 and 4.6, which indicates that at a collision, the momentum of the peakon and the anti-peakon goes to plus and minus infinity, respectively. Without conservational properties we are not guaranteed Lipschitz continuity on positions, momenta and shocks of the form in (4.16). Therefore, the argument in theorem 4.2 is not applicable for showing continuability past collisions for general multi-peakons. The multi-peakon dynamics is also very complex, questions we have to answer are:

- Do there exist collisions which involve more than one peakon and one anti-peakon, and if so, how do they behave?
- Can we after a while experience shockpeakon collisions. If so, how do they behave?

I have not been able to answer these questions, so I leave the problem of general multi-peakon solutions open. However, for a small subset of multi-peakons in which collisions do not occur, some rather obvious results have been achieved.

Theorem 4.7. *Assume $u = \sum_{i=1}^{n} m_i G_i$ is a multi-peakon fulfilling the following initial conditions*

- *The leftmost peakon lies to the right of the rightmost anti-peakon.*
- $\sum_{i=1}^{n} |m_i(0)| < \infty$.

Then collisions never occur, and the conservation property $\sum_{i=1}^{n} |m_i(t)| \leq \sum_{i=1}^{n} |m_i(0)|$ is true. Furthermore, the components $\{m_i\}$, $\{s_i\}$ and $\{x_i\}$ are Lipschitz continuous.

Proof. Since the multi-peakon is sorted position-wisely; $x_i(0) < x_{i+1}(0), \ \forall i$, there is an $n_1 \in \{2, \dots, n-1\}$ such that

$$m_i(0) < 0, \quad \forall i \in \{1, \dots, n_1\} \tag{4.43}$$

$$\text{and } m_i(0) > 0, \quad \forall i \in \{n_1 + 1, \dots, n\}. \tag{4.44}$$

Furthermore, from lemma 2.5(ii) we deduce that all anti-peakons will remain anti-peakons or vanish and all peakons will remain peakons or vanish. Hence it is valid to rewrite $u = \sum_{i=1}^{n_1} m_i^- G_i + \sum_{i=n_1+1}^{n} m_i^+ G_i$ where $m_i^- \leq 0$ and $m_i^+ \geq 0$. Let $M^- := \sum_{i=1}^{n_1} m_i^-$ and

$M^+ := \sum_{i=n_1+1}^{n} m_i^+$. Then we see, using the ODEs in (3.22), that

$$\begin{aligned}
\dot{M}^- &= \sum_{i=1}^{n_1} \dot{m}_i^- \\
&= \sum_{i=1}^{n_1} 2m_i^- \Big(\sum_{j=1}^{n_1} \operatorname{sign}(x_i - x_j) m_j^- G(x_i - x_j) - \sum_{j=n_1+1}^{n} m_i^+ G(x_i - x_j) \Big) \\
&= \sum_{i=1}^{n_1} -2m_i^- \Big(\sum_{j=n_1+1}^{n} m_i^+ G(x_i - x_j) \Big) \\
&\geq 0,
\end{aligned} \tag{4.45}$$

and similarly $\dot{M}^+ \leq 0$. Hence, all momenta are uniformly bounded in absolute value since

$$\begin{aligned}
|m_i^-(t)| \leq |M^-(t)| \leq |M^-(0)| \quad \forall i \in \{1, \ldots, n_1\} \\
|m_i^+(t)| \leq |M^+(t)| \leq |M^+(0)| \quad \forall i \in \{n_1 + 1, \ldots, n\}.
\end{aligned} \tag{4.46}$$

By lemma 2.5(iv) we know that if all momenta of u are uniformly bounded in absolute value then no collisions will occur. Lipschitz continuity follows from (4.16). □

Chapter 5

Constructible DP solutions

In [6] Coclite and Karlsen proved the following for the DP equation

Theorem 5.1. *Suppose $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then there exists an entropy weak solution to the Cauchy problem*

$$\begin{aligned} u_t - u_{xxt} + 4uu_x &= 3u_x u_{xx} + uu_{xxx}, \\ u(x,0) &= u_0(x). \end{aligned} \tag{5.1}$$

Fix any $T > 0$, and let $u, v : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be two entropy weak solutions of (5.1) *with initial data $u_0, v_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, respectively. Then for almost all $t \in (0, T)$*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \leq e^{M_T t} \|u_0 - v_0\|_{L^1(\mathbb{R})} \tag{5.2}$$

where

$$M_T := \frac{3}{2}\Big(\|u\|_{L^\infty(\mathbb{R}\times(0,T))} + \|v\|_{L^\infty(\mathbb{R}\times(0,T))} \Big) < \infty. \tag{5.3}$$

Consequently, there exists at most one entropy weak solution to (5.1).

Although every $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ has a unique weak entropy solution, finding it is not easy. Often one has to settle with approximating solutions numerically. Here multi-shockpeakons are interesting because we can base numerical schemes on their ODEs (3.22). But, as described in chapter 4, these schemes suffers at peakon collisions:

- We do not generally know how to continue solutions past collisions.
- Moment components $\{m_i\}$, or shock components $\{s_i\}$ can blow up in value at collisions.

However, if we restrict ourselves to the set of multi-shockpeakons with initial conditions

$$\mathscr{F} = \Big\{ u = \sum_{i=1}^{n} m_i G_i + s_i G_i' \Big| m_i(0) \geq s_i(0) \geq 0 \quad \text{and} \quad M(0) = \sum_{i=1}^{n} m_i(0) < \infty \Big\}, \tag{5.4}$$

we know how to explicitly continue the ODEs past collisions, and we know that all components are everywhere Lipschitz continuous. This is shown in theorem 4.2.
In this chapter we will show that functions in $\mathscr{F}$ are weak entropy solutions in $\mathbb{R} \times [0, T)$ for any $T > 0$. And, following that chain of thought, find the class of DP weak entropy solutions constructible by multi-shockpeakons from $\mathscr{F}$.

5.1 Properties of $\mathscr{F}$

Theorem 5.2. *If*

$$\sum_{i=1}^{n} m_i G_i + s_i G_i' = u \in \mathscr{F}, \quad M := \sum_{i=1}^{n} m_i(t), \quad S := \sum_{i=1}^{n} s_i(t) \tag{5.5}$$

,then

(i) $m_i(t) \geq s_i(t), \ \forall t \geq 0$

(ii) $u \geq 0$

(iii) $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$

(iv) $u \in L^\infty(\mathbb{R}_+; L^p(\mathbb{R})), \quad \forall p \in [1, \infty).$

(v) $u \in L^\infty(\mathbb{R}_+; BV(\mathbb{R}))$

(vi) $\|u(\cdot,t) - u(\cdot,w)\|_{L^1(\mathbb{R})} \leq 2M\Big(2(e^{8M^2|t-w|} - 1) + 5(1 - e^{-2M|t-w|})\Big).$

Proof. (i) This is proved in (4.14).

(ii)

$$u = \sum_{i=1}^{n} m_i G_i + s_i G_i' \geq \sum_{i=1}^{n} (m_i - s_i) G_i \geq 0 \tag{5.6}$$

(iii) Equation (3.47) shows that $\dot{M} = 0$. This implies that

$$\begin{aligned} u(x,t) &= \sum_{i=1}^{n} m_i G_i + s_i G_i' \\ &\leq 2M = 2M(0) \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+ \end{aligned} \tag{5.7}$$

(iv) We prove this in two turns. First $p = 1$, then the rest.

$$\begin{aligned} \|u(\cdot,t)\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} \sum_{i=1}^{n} m_i(t) G_i + s_i(t) G_i' dx \\ &= \sum_{i=1}^{n} \Big(\int_{-\infty}^{x_i(t)} (m_i(t) + s_i(t)) e^{x - x_i(t)} dx + \int_{x_i(t)}^{\infty} (m_i(t) - s_i(t)) e^{x_i(t) - x} dx \Big) \\ &= 2M < \infty \end{aligned} \tag{5.8}$$

For $p \in (1, \infty)$ we use the L^∞ boundedness from (iii)

$$\begin{aligned}
\|u(\cdot,t)\|^p_{L^p(\mathbb{R})} &= \int_{\mathbb{R}} \left(\sum_{i=1}^{n} m_i(t) G_i + s_i(t) G_i' \right)^p dx \\
&\leq \|u\|_\infty^{p-1} \sum_{i=1}^{n} \Big(\int_{-\infty}^{x_i(t)} (m_i(t) + s_i(t)) e^{x - x_i(t)} dx + \int_{x_i(t)}^{\infty} (m_i(t) - s_i(t)) e^{x_i(t) - x} dx \Big) \\
&= (2M)^p < \infty
\end{aligned} \tag{5.9}$$

(v) Using the definition of bounded variation, definition B.1, yields

$$\begin{aligned}
\|u(\cdot,t)\|_{BV(\mathbb{R})} &\leq \int_{\mathbb{R}} |u| + |D_x u| dx \\
&= 2M + \int_{\mathbb{R}} \Big| \sum_{i=1}^{n} m_i G_i' + s_i G_i + 2 s_i \delta_{x_i} \Big| dx \\
&\leq 2M + \sum_{i=1}^{n} \int_{\mathbb{R}} m_i G_i + s_i G_i + 2 s_i \delta_{x_i} dx \\
&= 4 \sum_{i=1}^{n} m_i + s_i \\
&\leq 4 \|u(\cdot,0)\|_1
\end{aligned} \tag{5.10}$$

(vi) Straight forward, but long! See appendix C.

□

Given the mapping $T_0 : Ł^\infty(\mathbb{R} \times [0,T)) \to Ł^\infty(\mathbb{R})$ defined by $T_0(u) = u(x,0)$. Set $\mathscr{F}(0) := T_0(\mathscr{F})$. That is,

$$u_0 \in \mathscr{F}(0) \iff \begin{cases} u_0(x) = \sum_{i=1}^{n} m_i(0) G_i(x - x_i(0)) + s_i(0) G_i'(x - x_i(0)), \\ m_i(0) \geq s_i(0) \;\; \forall i, \text{ and } \sum_{i=1}^{n} m_i(0) < \infty. \end{cases} \tag{5.11}$$

Theorem 5.3. *If $u_0 \in \mathscr{F}(0)$, then its multi-shockpeakon continuation $u \in \mathbb{R} \times [0,T)$ is the unique weak DP entropy solution to the Cauchy problem*

$$\begin{aligned}
u_t - u_{xxt} + 4 u u_x &= 3 u_x u_{xx} + u u_{xxx}, \\
u(x,0) &= u_0(x),
\end{aligned} \tag{5.12}$$

for any $T > 0$.

Proof. By definition, u is an element of $\mathscr{F}$. The boundedness criterion

$$u \in L^\infty(\mathbb{R} \times [0,T)) \cap L^\infty([0,T); L^2(\mathbb{R})) \cap L^\infty((0,T); BV(\mathbb{R})) \tag{5.13}$$

then follows from theorem 5.2(iii),(iv) and (v).
When proving that the weak solution condition (3.8) is fulfilled, collisions are an obstacle. Theorem 3.7 assures that multi-shockpeakons satisfy this condition, but in its proof, it is assumed that all shockpeakons are position-wise distinct ($x_i \neq x_{i+1} \quad \forall i$). At collisions this is of course not true. However, the way multi-shockpeakon solutions of $\mathscr{F}$ are continued past collisions (see theorem 4.2) implies that all shockpeakons are locally distinct; only at collision times are some of the shockpeakons bound to be non-distinct. If a multi-shockpeakon initially has n shockpeakons, then we know, from the fact that at a collision two or more shockpeakons "melt" into one shockpeakon, that there are at most n times for which the entropy condition is not satisfied. And from theorem 5.2 (vi) we know that the following condition

$$\|u(\cdot,t^-) - u(\cdot,t^+)\|_1 = 0, \quad \forall t \in (0,T) \tag{5.14}$$

holds (also at collision times). The fact that condition (3.8) is satisfied everywhere except on a set of measure zero in $\mathbb{R} \times [0,T)$ combined with equation (5.14), proves that condition (3.8) is fulfilled.
An argument of the same type proves that the entropy condition holds.

By theorem 5.2(iii) and (v) we see that $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, and then, by theorem 5.1, u is the unique weak entropy solution to the Cauchy problem (5.12) ☐

5.2 $\mathscr{F}$ constructible DP solutions

Based on theorem 5.2 we know that if $u_0 \in \mathscr{F}(0)$, then its unique weak entropy solution u lies in $\mathscr{F}$. So it is trivially constructible by multi-shockpeakons in $\mathscr{F}$. However, the space of $\mathscr{F}$ constructible solutions is bigger. Here we will study the set of functions

$$\mathscr{H} = \left\{ f \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) \middle| \langle f, \phi \rangle \geq 0 \text{ and } \langle Df - f, \phi \rangle \leq 0,\ \forall \phi \in \mathcal{D}(\mathbb{R}),\ \text{such that } \phi \geq 0 \right\}, \tag{5.15}$$

where

$$\langle f, \phi \rangle = \int_{\mathbb{R}} f\phi \, dx. \tag{5.16}$$

Theorem 5.4. *Suppose $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then u_0's unique entropy solution to the Cauchy problem (5.1), u, is $\mathscr{F}$ constructible if and only if $u_0 \in \mathscr{H}$*

Proof. Proposition 5.5, 5.6 and 5.8. ☐

The proof of this theorem is divided into four steps. First, examining the sequence of functions $\{u_{0,n}\}$ going to u_0. Second, obtaining $\mathscr{F}$ compactness such that the multi-shockpeakon continuation series of $\{u_{0,n}\}$, $\{u_n\}$, converges to some u. Third, proving that u is an entropy solution. And fourth, proving the only if statement; if a DP solution u with initial function $u_0 \in L^1 \cap BV$ is $\mathscr{F}$ constructible, then $u_0 \in \mathscr{H}$.

Proposition 5.5. *Suppose $u_0 \in \mathscr{H}$, then there exists a sequence $\{u_{0,n}\} \subset \mathscr{F}(0)$ such that*

$$\begin{aligned} u_{0,n} \to u_0, \quad & \text{in } L^1(\mathbb{R}), \\ \|u_{0,n}\|_1 \leq \|u_0\|_1 \quad & \forall n \in \mathbb{N} \\ \text{and } \|u_{0,n}\|_{BV(\mathbb{R})} \leq 4\|u_0\|_1 \quad & \forall n \in \mathbb{N}. \end{aligned} \tag{5.17}$$

Proof. We begin by mollifying u_0. Let

$$\rho(x) := \begin{cases} Ce^{\frac{1}{1-x^2}}, & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases} \tag{5.18}$$

The constant C is chosen such that

$$\int_{\mathbb{R}} \rho(x)\, dx = 1. \tag{5.19}$$

Then u_0 is mollified by $u_0^\varepsilon = \rho_\varepsilon * u_0$ where $\rho_\varepsilon(x) = (1/\varepsilon)\rho(x/\varepsilon)$. The mollified function converges to u_0 in L^1 as ε goes to zero. Furthermore, u_0^ε is smooth, non-negative and

$$(u_0^\varepsilon)_x = \rho_\varepsilon * Du_0(x) \leq \rho_\varepsilon * u_0(x) = u_0^\varepsilon(x). \tag{5.20}$$

In particular, (5.20) implies that

$$u_0^\varepsilon(x) \geq u^\varepsilon(y)e^{x-y}, \quad \forall x < y. \tag{5.21}$$

We prove the convergence in two steps. First, we show that for each $\varepsilon > 0$ we can find a sequence of functions $\{u_{0,\varepsilon,n}\}_n \subset \mathscr{H}$ converging to u_0^ε in L^1. Second, we pick a sequence $\{u_{0,k}\}_k \subset \{u_{0,\varepsilon,n}\}_{\varepsilon,n}$ that converges to u_0 in L^1.

We choose a sequence $\{u_{0,\varepsilon,n}\} \subset \mathscr{F}(0)$ with shocks equal to momenta

$$u_{0,\varepsilon,n} = \sum_{i=1}^{n^2} m_i G_i(x - x_i) + m_i G_i'(x - x_i), \tag{5.22}$$

and determine the shockpeakon values $\{x_i, m_i, m_i\}_{i=1}^{n^2}$ in $u_{0,\varepsilon,n}$ starting with the n^2th

$$\begin{aligned} x_i &= \frac{2n(i-1)}{n^2-1} - n, \quad i \in \{1, 2, ..., n^2\} \\ s_{n^2} = m_{n^2} &= \frac{u_0^\varepsilon(x_{n^2})}{2} \\ s_i = m_i &= \frac{u_0^\varepsilon(x_i) - \sum_{j=i+1}^{n^2} m_j(G(x_j - x_i) + G'(x_j - x_i))}{2}, \quad i \in \{1, 2, \ldots, n^2 - 1\} \end{aligned} \tag{5.23}$$

One thing needs justification in the determination scheme; shocks must be non-negative.

Beginning with the last shock, s_{n^2}, we see it is non-negative since u_0^ε is non-negative. Using (5.21) we find that

$$u_0^\varepsilon(x) - m_{n^2}(G(x - x_{n^2}) + G'(x - x_{n^2})) = \begin{cases} u_0^\varepsilon(x) \geq 0 & x > x_n \\ \frac{u_0^\varepsilon(x_{n^2})}{2} \geq 0 & x = x_{n^2} \\ u_0^\varepsilon(x) - u_0^\varepsilon(x_{n^2}^-)e^{x - x_{n^2}^-} \geq 0 & x < x_{n^2} \end{cases} \tag{5.24}$$

Hence

$$2s_{n^2-1} = u_0^\varepsilon(x_{n^2-1}) - m_{n^2}(G(x_{n^2-1} - x_{n^2}) + G'(x_{n^2-1} - x_{n^2})) \geq 0. \tag{5.25}$$

Define

$$u^1_{0,\varepsilon,n} := u_0^\varepsilon \chi_{(-\infty, x_{n^2})} - m_{n^2}(G_{n^2} + G'_{n^2}). \tag{5.26}$$

Then $u^1_{0,\varepsilon,n}$ is non-negative everywhere and smooth on $(-\infty, x_{n^2})$. Similarly to (5.24) we get

$$u^1_{0,\varepsilon,n} - m_{n^2-1}(G_{n^2-1} + G'_{n^2-1}) \geq 0. \tag{5.27}$$

We see that s_{n^2-2} is non-negative by the inequality

$$u_0^\varepsilon - \sum_{j=n^2+1-2}^{n} m_j(Gj + G'j) = u_0^\varepsilon \chi_{[x_{n^2}, \infty)} + u^1_{0,\varepsilon,n} - m_{n^2-1}(G_{n^2-1} + G'_{n^2-1}) \geq 0. \tag{5.28}$$

Inductively we construct functions

$$u^i_{0,\varepsilon,n} := u^{i-1}_{0,\varepsilon,n} \chi_{(-\infty, x_{n^2+1-i})} - m_{n^2+1-i}(G_{n^2+1-i} + G'_{n^2+1-i}) \quad i = 2, ..., n^2. \tag{5.29}$$

These functions affirm that shocks are non-negative

$$\begin{aligned}
2s_{n^2-i} &= u_0^\varepsilon - \sum_{j=n^2+1-i}^{n^2} m_j(G_j + G'_j) \\
&\geq u^1_{0,\varepsilon,n} - \sum_{j=n^2+1-i}^{n^2-1} m_j(G_j + G'_j) \\
&\geq \cdots \\
&\geq u^{i-1}_{0,\varepsilon,n} - m_{n^2-i}(G_{n^2-i} + G'_{n^2-i}) \\
&\geq u^i_{0,\varepsilon,n} \\
&\geq 0 \qquad \forall i \in \{1, 2, ..., n^2 - 1\}.
\end{aligned} \tag{5.30}$$

By (5.29) we can also show that $(u_{0,\varepsilon,n})_{n\in\mathbb{N}}$ is bounded by u_0^ε;

$$u_0^\varepsilon - u_{0,\varepsilon,n} \geq u^{n^2}_{0,\varepsilon,n} \geq 0 \quad \forall n \in \mathbb{N}. \tag{5.31}$$

Theorem 5.2 and this boundedness implies that all $u_{0,\varepsilon,n}$ are uniformly bounded in $L^1 \cap BV$:

$$\begin{aligned}
\|u_{0,\varepsilon,n}\|_1 \leq \|u_0^\varepsilon\|_1 \leq \|u_0\|_1 \qquad \forall n \in \mathbb{N}, \\
\|u_{0,\varepsilon,n}\|_{BV(\mathbb{R})} \leq 4\|u_{0,\varepsilon,n}\|_1 \leq 4\|u_0\|_1 \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{5.32}$$

The convergence $u_{0,\varepsilon,n} \to u_0^\varepsilon$ is proved as follows. Given $\epsilon > 0$ there exists an $R \in \mathbb{R}$ such that $\|u_0^\varepsilon(1 - \chi_{(-R,R)})\|_{L^1} \leq \frac{\epsilon}{2}$. Let $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceil operation, respectively. Furthermore, notice that $\|u_0^\varepsilon\|_\infty \leq \frac{1}{2}\|u_0^\varepsilon\|_{BV(\mathbb{R})}$ and

$$\max_{x\in(x_i, x_{i+1})} u_0^\varepsilon(x_{i+1}) - u_{0,\varepsilon,n}(x) \leq u_0^\varepsilon(x_{i+1})(1 - e^{x_i - x_{i+1}}). \tag{5.33}$$

With this in mind we investigate the L^1 norm of $(u_0^\varepsilon - u_{0,\varepsilon,n})\chi_{(-R,R)}$ for $n \geq R$

$$\begin{aligned}
\|(u_0^\varepsilon - u_{0,\varepsilon,n})\chi_{(-R,R)}\|_{L^1} &= \sum_{i=\lfloor \frac{(n-R)(n^2-1)}{2n} \rfloor +1}^{\lceil \frac{(R+n)(n^2-1)}{2n} \rceil} \int_{(x_i,x_{i+1})} u_0^\varepsilon(x) - u_{0,\varepsilon,n}(x)dx \\
&\leq \sum_{i=\lfloor \frac{(n-R)(n^2-1)}{2n} \rfloor +1}^{\lceil \frac{(R+n)(n^2-1)}{2n} \rceil} \int_{(x_i,x_{i+1})} essV_{x_i}^{x_{i+1}}(u_0^\varepsilon) + u_0^\varepsilon(x_{i+1}) - u_{0,\varepsilon,n}(x)\, dx \\
&\leq \|u_0^\varepsilon\|_{BV(\mathbb{R})} \frac{2n}{n^2-1} + (2R + \frac{3n}{n^2-1})(1 - e^{-\frac{2n}{n^2-1}})\|u_0^\varepsilon\|_\infty \\
&\leq \|u_0^\varepsilon\|_{BV(\mathbb{R})} \Big(\frac{2n}{n^2-1} + (R + \frac{3n}{n^2-1})(1 - e^{-\frac{2n}{n^2-1}}) \Big) \to 0 \\
&\text{as } n \to \infty.
\end{aligned} \tag{5.34}$$

Therefore, for any $R \in \mathbb{R}$ and $\epsilon > 0$ there exists an $N(R,\epsilon) \in \mathbb{N}$ such that

$$\|(u_0^\varepsilon - u_{0,\varepsilon,n})\chi_{(-R,R)}\|_{L^1} \leq \frac{\epsilon}{2}, \quad \text{for all } n > N(R,\epsilon). \tag{5.35}$$

This yields

$$\begin{aligned}
\|(u_0^\varepsilon - u_{0,\varepsilon,n})\|_{L^1} &\leq \|(u_0^\varepsilon - u_{0,\varepsilon,n})(1-\chi_{(-R,R)})\|_{L^1} + \|(u_0^\varepsilon - u_{0,\varepsilon,n})\chi_{(-R,R)}\|_{L^1} \\
&\leq \|u_0^\varepsilon(1-\chi_{(-R,R)})\|_{L^1} + \|(u_0^\varepsilon - u_{0,\varepsilon,n})\chi_{(-R,R)}\|_{L^1} \\
&\leq \epsilon, \qquad \text{for all } n > N(R,\epsilon)
\end{aligned} \tag{5.36}$$

To prove general convergence, define the sequence $\{u_{0,n}\}_n$ with

$$\begin{aligned}
u_{0,n} &= u_{0,\varepsilon_n,h(n)} \\
\text{where } \varepsilon_n &= \frac{1}{n} \text{ and } h(n) := \min\Big\{ m \in \mathbb{N} \Big| \|u_0^{\varepsilon_n} - u_{0,\varepsilon_n,m}\|_1 \leq \frac{1}{n} \Big\}.
\end{aligned} \tag{5.37}$$

Given $\epsilon > 0$ there exists an $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned}
\|u_0 - u_0^{\varepsilon_n}\|_1 &\leq \frac{\epsilon}{2}, \quad \text{for all } n > N_1, \\
\|u_0^{\varepsilon_n} - u_{0,n}\|_1 &\leq \frac{\epsilon}{2} \quad \text{for all } n > N_2.
\end{aligned} \tag{5.38}$$

Thus,

$$\begin{aligned}
\|u_0 - u_{0,n}\|_1 &\leq \|u_0 - u_0^{\varepsilon_n}\|_1 + \|u_0^{\varepsilon_n} - u_{0,n}\|_1 \\
&\leq \epsilon, \qquad \text{for all } n > \max(N_1, N_2).
\end{aligned} \tag{5.39}$$

□

Proposition 5.6. *Suppose $\{u_{0,n}\} \subset \mathscr{F}(0)$ is a sequence converging to u_0 as described in proposition 5.5. Then for any $T > 0$, the corresponding entropy solutions of $\{u_{0,n}\}$, $\{u_n\} \subset \mathscr{F}$, converge to a function $u \in L^\infty(\mathbb{R} \times [0,T))$ in the following sense*

$$u_n \to u \quad \text{in } L^p(\mathbb{R} \times [0,T)), \quad \forall p \in [1,\infty). \tag{5.40}$$

Furthermore, u is the unique entropy solution to the Cauchy problem (5.1).

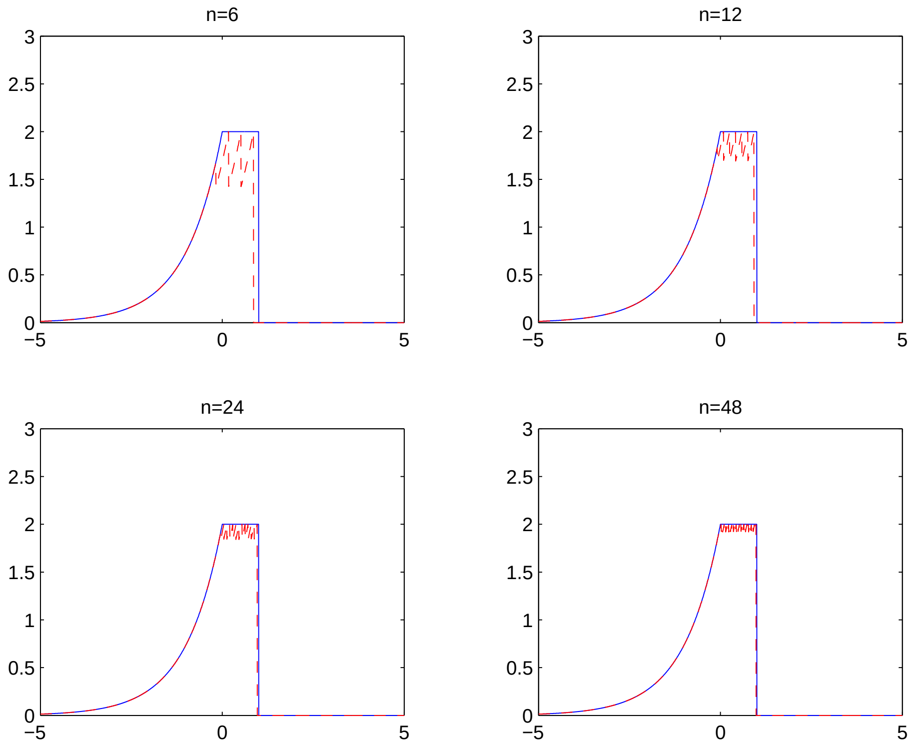

Figure 5.1: The sequence $\{u_{0,n}\}$ from (5.17) converging to $f(x) = 2e^x\chi_{(-\infty,0]}(x) + 2\chi_{(0,1]}(x)$.

Alternative proof. The Coclite and Karlsen criterion that if u is the unique entropy weak solution corresponding to u_0, then for almost all $t \in (0, T)$

$$\|u(\cdot,t) - u_n(\cdot,t)\|_{L^1(\mathbb{R})} \leq e^{M_T^n t}\|u(\cdot,0) - u_{0,n}\|_{L^1(\mathbb{R})},$$

where

$$M_T^n = \frac{3}{2}\Big(\|u\|_{L^\infty(\mathbb{R}\times[0,T))} + \|u_n\|_{L^\infty(\mathbb{R}\times[0,T))}\Big) \leq C(T), \quad \forall n \in \mathbb{N}.$$

Since $u_{0,n} \to u_0$ in L^1, this implies that $u_n \to u$ in $L^p(\mathbb{R} \times [0,T))$ $\forall p \in [1,\infty)$. □

Remark 5.7. The alternative proof is much shorter and in most ways better than the original one, but the interesting property $\|u\|_{L^\infty(\mathbb{R}\times[0,T))} \leq \|u(\cdot,0)\|_1$, which is shown in the original proof, is not shown in the alternative one.

Original proof. We start by showing that for each $t \in [0,T)$ there exists a sequence $\{u_{n_i}(t)\}$ such that $u_{n_i}(t)$ converges to a function $u(t)$ in L^1. To do this we use Kolmogorov's compactness theorem B.3 with $\Omega = \mathbb{R}$ and $p = 1$. For each $t \in [0,T)$ we verify that conditions (i), (ii) and (iii) of Kolmogorov's theorem are fulfilled.
For a given multi-shockpeakon $u_{0,n} = \sum_{i=1}^{n^2} m_i G_i + s_i G_i'$ define M_n to be the sum of its momenta:

$$M_n := \sum_{i=1}^{n^2} m_i \tag{5.41}$$

From theorem 5.2 and proposition 5.5 we deduce these bounds

$$\begin{aligned} \|u_n(\cdot,t)\|_{L^1} &\le 2M_n \le \|u_{0,n}\|_{L^1} \le \|u_0\|_{L^1} && \forall n \in \mathbb{N},\ t\in[0,T], \\ \|u_n(\cdot,t)\|_{BV(\mathbb{R})} &\le 8M_n \le 4\|u_0\|_{L^1} && \forall n \in \mathbb{N},\ t\in[0,T], \\ \|u_n\|_{L^\infty(\mathbb{R}\times[0,T))} &\le 2M_n \le \|u_0\|_{L^1} && \forall n \in \mathbb{N}. \end{aligned} \tag{5.42}$$

Condition (i) is obviously fulfilled by this.

Using ess $V_{-\infty}^{\infty}(u_n)$ (defined in B.1), and (5.42), we see that

$$\begin{aligned} \int_{\mathbb{R}} |u_n(x+\varepsilon,t)-u_n(x,t)|\,dx &= \sum_{j\in\mathbb{Z}} \int_{(j-1)\varepsilon}^{j\varepsilon} |u_n(x+\varepsilon,t)-u_n(x,t)|\,dx \\ &= \int_0^{|\varepsilon|} \sum_{j\in\mathbb{Z}} |u_n(x+j\varepsilon,t)-u_n(x+(j-1)\varepsilon,t)|\,dx \\ &\le \int_0^{|\varepsilon|} \text{ess } V_{-\infty}^{\infty}(u_n)\,dx \\ &\le |\varepsilon|\|u_n(\cdot,t)\|_{BV(\mathbb{R})} \end{aligned} \tag{5.43}$$

which implies that condition (ii) is fulfilled

$$\|u_n(\cdot+\varepsilon,t)-u_n(\cdot,t)\| \le 4|\varepsilon|\|u_0\|_1, \quad \forall n\in\mathbb{N}, \forall t\in[0,T). \tag{5.44}$$

Condition (iii)

$$\lim_{\alpha\to\infty} \int_{\{x\in\mathbb{R}\,|\,|x|\ge\alpha\}} |u_n(x,t)|\,dx = 0 \text{ uniformly for all } u_n, \text{ and } t\in[0,T], \tag{5.45}$$

is more difficult to prove than the previous ones. Given $\varepsilon>0$ we know that there exists an $R_1 \in \mathbb{R}_+$ such that

$$\|u_0(1-\chi_{(-R_1,R_1)})\|_{L^1} < \varepsilon. \tag{5.46}$$

From (5.31) and the definition of $u_{0,n}$ in equation (5.37) we deduce that

$$\begin{aligned} \|u_{0,n}(1-\chi_{(-R_1-1,R_1+1)})\|_{L^1} &\le \|u_0^{\varepsilon_n}(1-\chi_{(-R_1-1,R_1+1)})\|_{L^1} \\ &\le \|u_0(1-\chi_{(-R_1,R_1)})\|_{L^1} \\ &\le \varepsilon \qquad \forall n\in\mathbb{N} \end{aligned} \tag{5.47}$$

This implies that for all $u_{0,n}$ the sum of the momenta from shockpeakons outside of $(-R_1-1, R_1+1+\ln 2)$ is less than ε. We set $R_2 = R_1+1+\ln 2$, and phrase this mathematically:

$$\sum_{i\in\mathcal{A}(u_{0,n})} m_i < \varepsilon, \quad \forall u_{0,n}, \tag{5.48}$$

where

$$\mathcal{A}(u_{0,n}) := \{i \in \{1,2,\dots,n^2\} \mid |x_i| \ge R_2\}. \tag{5.49}$$

From (3.22) and (5.42) we deduce position movement and momentum growth properties for all the shockpeakons in all u_n functions

$$\begin{aligned} 0 \le \dot{x}_i \le \frac{\|u_o\|_1}{2}, &\implies x(0) \le x_i(t) \le x(0) + \frac{\|u_0\|_1}{2}t \\ \dot{m}_i \le 4m_i\|u_0\|_1 &\implies m_i(t) \le m_i(0)e^{4\|u_0\|_1 t} \end{aligned} \tag{5.50}$$

Giving a growth bound on shockpeakons outside of $(-R_2, R_2)$:

$$\sum_{i\in\mathcal{A}(u_{0,n})} m_i(t) < \varepsilon e^{4\|u_0\|_1 t}, \quad \forall n \in \mathbb{N} \tag{5.51}$$

Furthermore, for a given time t, shockpeakons of u_n initially inside $(-R_2, R_2)$ are, because of the position movement property, inside $(-R_2, R_2 + (\|u_0\|_1/2)t)$. Thus, the set of shockpeakons outside $(-R_2, R_2 + (\|u_0\|_1/2)t)$ must be a subset of $\{x_i, m_i, s_i\}_{i\in\mathcal{A}(u_{0,n})}$. We set

$$R_3 = R_2 + \frac{\|u_0\|_1}{2}T + \ln(\frac{\|u_0\|_1}{\varepsilon}), \tag{5.52}$$

and take the L^1 norm of $u_n(\cdot,t)(1-\chi_{(-R_3,R_3)})$:

$$\begin{aligned}\int_{\{x\in\mathbb{R}||x|\geq R_3\}} |u_n(x,t)|\, dx &< \int_{\{x\in\mathbb{R}||x|\geq R_3\}} \|u_0\|_1 e^{-ln(\frac{\|u_0\|_1}{\varepsilon})-x}\, dx + \sum_{i\in\mathcal{A}(u_{0,n})} 2m_i(t) \\ &< \int_{\{x\in\mathbb{R}||x|\geq R_3\}} \varepsilon e^{-|x|} dx + 2\varepsilon e^{4\|u_0\|_1 T} \\ &< 2\varepsilon(1+e^{4\|u_0\|_1 T}), \quad \forall n \in \mathbb{N} \text{ and } t \in [0,T].\end{aligned} \tag{5.53}$$

Now, given an $\epsilon > 0$ we confirm condition (iii) by following the argument above with $\varepsilon = \epsilon/(2+2e^{4\|u_0\|_1 T})$:

$$\int_{\{x\in\mathbb{R}||x|\geq R_3(\varepsilon)\}} |u_n(x,t)| dx < \varepsilon(2+2e^{4\|u_0\|_1 T}) = \epsilon, \quad \forall n \in \mathbb{N} \text{ and } t \in [0,T], \tag{5.54}$$

Consequently, for any $T > 0$ and each $t \in [0,T)$, $\{u_n(t)\}_n$ has a convergent subsequence (still denoted by n) $\{u_n(t)\}_n$ such that $u_n(t) \to u(t)$ in $L^1(\mathbb{R})$.

The next step is to extend convergence to $L^1(\mathbb{R}\times[0,T))$. Deduce from theorem 5.2(vii) and (5.42) that

$$\|u_n(\cdot,t)-u_n(\cdot,w)\|_1 \leq 2\|u_0\|_1\Big(2(e^{8\|u_0\|_1^2|t-w|}-1)+5(1-e^{-2\|u_0\|_1|t-w|})\Big), \quad \forall t \in \mathbb{R}_+, \forall n \in \mathbb{N} \tag{5.55}$$

Take a dense countable subset of the interval $[0,T)$, $E = [0,T)\cap\mathbb{Q}$. By a diagonal argument on $\{u_n\}$ we obtain a convergent subsequence $\{u_{n_k}\}$:

$$\|u(\cdot,t)-u_{n_k}(\cdot,t)\|_1 \to 0 \quad \text{as } n_k \to \infty, \quad \forall t \in E. \tag{5.56}$$

Assume $\varepsilon > 0$ is given. Then by (5.55) there exists a $\delta(\varepsilon) > 0$ such that

$$\|u_{n_k}(\cdot,t)-u_{n_k}(\cdot,w)\|_1 \leq \varepsilon, \quad \forall n_k,\ \forall t \in [0,T), \text{ when } |t-w| < \delta(\varepsilon). \tag{5.57}$$

For any t in $[0,T)$ we find a $t_i \in E$ fulfilling $|t-t_i| < \delta(\varepsilon)$. Thereby

$$\|u_{n_k}(\cdot,t)-u_{n_k}(\cdot,t_i)\|_1 \leq \varepsilon, \quad \forall n_k. \tag{5.58}$$

We also have that $\{u_{n_k}\}$ is Cauchy on E. Expressed with respect to ε as

$$\|u_{n_j}(\cdot,t_i)-u_{n_k}(\cdot,t_i)\|_1 \leq \varepsilon, \quad \forall n_j, n_k \geq N, \quad \forall t_i \in E. \tag{5.59}$$

Using these results and the triangle inequality, we can show that $\{u_{n_k}\}$ is Cauchy for every $t \in [0, T)$:

$$\begin{aligned}\|u_{n_j}(\cdot,t) - u_{n_k}(\cdot,t)\|_1 \leq \|u_{n_j}(\cdot,t) - u_{n_j}(\cdot,t_i)\|_1 + \|u_{n_j}(\cdot,t_i) - u_{n_k}(\cdot,t_i)\|_1 \\ + \|u_{n_k}(\cdot,t_i) - u_{n_k}(\cdot,t)\|_1 \\ \leq 3\varepsilon.\end{aligned} \tag{5.60}$$

L^1 is a Banach space, which implies that for each $t \in [0, T)$ we have $u_{n_k}(t) \to u(t)$ in L^1. Define $g_{n_k}(t) := \|u(\cdot,t) - u_{n_k}(\cdot,t)\|_1$. Then $g_{n_k}(t)$ converges point-wise to 0 in $[0, T)$ and is bounded:

$$\begin{aligned}|g_{n_k}(t)| &\leq \|u(\cdot,t)\|_1 + \|u_{n_k}(\cdot,t)\|_1 \\ &\leq \underset{n_k}{sup}\|u_{n_k}(\cdot,t)\|_1 + \|u_0\|_1 \\ &\leq 2\|u_0\|_1, \quad \forall n_k, \quad \forall t \in [0, T).\end{aligned} \tag{5.61}$$

Lebesgue's dominated convergence theorem then gives $g_{n_k} \to 0$ in $L^1([0, T))$, which implies $u_{n_k} \to u$ in $L^1(\mathbb{R} \times [0, T))$. The convergence also means that u is bounded in L^∞ norm

$$\|u\|_{L^\infty(\mathbb{R}\times[0,T))} \leq \sup_{n_k} \|u_{n_k}\|_{L^\infty(\mathbb{R}\times[0,T))} \leq \|u_0\|_1, \tag{5.62}$$

which makes it easy to show that the convergence is more general

$$\begin{aligned}\|u - u_{n_k}\|^p_{L^p(\mathbb{R}\times[0,T))} &= \iint_{\mathbb{R}\times[0,T)} |u - u_{n_k}|^p \, dx \, dt \\ &\leq \|u\|^{p-1}_{L^\infty(\mathbb{R}\times[0,T))} \iint_{\mathbb{R}\times[0,T)} |u - u_{n_k}| \, dx \, dt \to 0 \\ &\implies u_{n_k} \to u \text{ in } L^p(\mathbb{R} \times [0, T)), \quad \forall p \in [1, \infty).\end{aligned} \tag{5.63}$$

Having proved the convergence part, what is left is proving that u is an entropy solution. For u to be an entropy solution it has to satisfy conditions in definition 3.3. From the first part of this proof we see that $u \in L^\infty(\mathbb{R} \times [0, T)) \cap L^\infty([0, T); L^2(\mathbb{R}))$. By theorem B.2 and the BV bound in equation (5.42) we have

$$\|D_x u(\cdot,t)\|(\mathbb{R}) \leq \liminf_{n_k\to\infty} \|D_x u_{n_k}(\cdot,t)\|(\mathbb{R}) \leq 4\|u_0\|_1, \quad \text{a.e. on } [0, T). \tag{5.64}$$

Hence $u \in L^\infty([0, T); BV(\mathbb{R}))$, and all boundedness criterions are fulfilled.

What then remains is that u fulfils the weak solution criterion (3.8) and the entropy criterion (3.10).

Looking at (3.8) we need

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \phi_t u + \phi_x \frac{1}{2}u^2 - \phi D_x P^u \, dxdt + \int_{\mathbb{R}} \phi(x,0)u(x,0)dx = 0, \\ \forall \phi \in C_c^\infty(\mathbb{R} \times [0, T)), \\ \text{where } P^u(x,t) := \int_{\mathbb{R}} \frac{3}{4} e^{-|x-y|} u(y,t)^2 dy.\end{aligned} \tag{5.65}$$

Theorem 5.3 tells us that all u_{n_k} functions are entropy solutions, and thereby fulfil (3.8) and (3.10). Therefore

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \phi_t u + \phi_x \frac{1}{2} u^2 - \phi D_x P^u \, dxdt + \int_{\mathbb{R}} \phi(x,0) u_0 dx \\
&= \int_0^T \int_{\mathbb{R}} \phi_t (u - u_{n_k}) + \phi_x \frac{1}{2}(u^2 - u_{n_k}^2) - \phi(D_x P^u - D_x P^{u_{n_k}}) \, dxdt + \int_{\mathbb{R}} \phi(x,0)(u_0 - u_{0,n_k}) dx.
\end{aligned} \tag{5.66}$$

Strong convergence implies weak convergence, so the first integrand vanishes

$$\int_0^T \int_{\mathbb{R}} \phi_t (u - u_{0,n_k}) \, dxdt \to 0 \quad \text{as } n_k \to \infty, \tag{5.67}$$

and, likewise, the last integrand, $\phi(x,0)(u_0 - u_{0,n_k})$, vanishes.

Since all functions are L^∞ bounded, the second integrand also vanishes

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} \phi_x \frac{1}{2}(u^2 - u_{n_k}^2)) \, dxdt &\leq \frac{1}{2}(\|u\|_\infty + \|u_{n_k}\|_\infty) \|\phi_x\|_\infty \int_0^T \int_{\mathbb{R}} |u - u_{n_k}| \, dxdt \\
&\leq C \int_0^T \int_{\mathbb{R}} |u - u_{n_k}| \, dxdt \to 0 \quad \text{as } n_k \to \infty.
\end{aligned} \tag{5.68}$$

Applying Fubini's theorem and boundedness, the third integrand vanishes

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} \phi(D_x P^u - D_x P^{u_{n_k}}) \, dx \, dt &= \int_0^T \int_{\mathbb{R}} \phi \int_{\mathbb{R}} \frac{3}{4} - \text{sign}(x-y) e^{-|x-y|} (u(y,t)^2 - u_{n_k}(y,t)^2) \, dy \, dx \, dt \\
&\leq 2\|u_0\|_1 \|\phi_x\|_\infty \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-|x-y|} |u(y,t) - u_{n_k}(y,t)| \, dy \, dxdt \\
&= C \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-|x-y|} |u(y,t) - u_{n_k}(y,t)| \, dx \, dy \, dt \\
&= 2C \int_0^T \int_{\mathbb{R}} |u(y,t) - u_{n_k}(y,t)| \, dy \, dt \\
&\to 0 \quad \text{as } n_k \to \infty
\end{aligned} \tag{5.69}$$

Hence (3.8) is fulfilled.

The entropy condition (3.10) is:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \eta(u)\phi_t + q(u)\phi_x - \eta'(u) D_x P^u \phi \, dx \, dt + \int_{\mathbb{R}} \eta(u(x,0))\phi(x,0) \, dx \geq 0, \\
&\qquad \forall \phi \in C_c^\infty(\mathbb{R} \times [0,T)), \text{ such that } \phi \geq 0,
\end{aligned} \tag{5.70}$$

for any convex C^2 entropy $\eta : \mathbb{R} \to \mathbb{R}$ with corresponding entropy flux $q'(u) = \eta'(u)u$.

All u_{n_k} functions satisfy this condition, therefore

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \eta(u)\phi_t + q(u)\phi_x - \eta'(u) D_x P^u \phi \, dx \, dt + \int_{\mathbb{R}} \eta(u(x,0))\phi(x,0) \, dx \\
&\geq \int_0^T \int_{\mathbb{R}} \Big((\eta(u) - \eta(u_{n_k}))\phi_t + (q(u) - q(u_{n_k}))\phi_x \\
&\qquad - (\eta'(u) D_x P^u - \eta'(u_{n_k}) D_x P^{u_{n_k}})\phi \Big) \, dx \, dt \\
&\quad + \int_{\mathbb{R}} \Big(\eta(u(x,0)) - \eta(u_{n_k}(x,0)) \Big) \phi(x,0) \, dx.
\end{aligned} \tag{5.71}$$

Since $\eta \in C^2$ and

$$\operatorname{supp}(u), \operatorname{supp}(u_{n_k}) \subseteq \Big[0,\ \|u_0\|_1\Big], \tag{5.72}$$

we have local Lipschitz continuity for η, η' and q:
Let $C_\eta := \max_{\zeta\in[0,\|u_0\|_1]}(\eta(\zeta))$ and $C_{\eta'} := \max_{\zeta'\in[0,\|u_0\|_1]}(\eta'(\zeta))$, then

$$\left.\begin{array}{l} |\eta(\zeta) - \eta(\zeta')| \leq C_\eta|\zeta - \zeta'|, \\ |\eta'(\zeta) - \eta'(\zeta')| \leq C_{\eta'}|\zeta - \zeta'|, \\ |q(\zeta) - q(\zeta')| \leq \|u_0\|_1 C_{\eta'}|\zeta - \zeta'|, \end{array}\right\} \text{ for } 0 \leq \zeta, \zeta' \leq \|u_0\|_1. \tag{5.73}$$

We look at the integrand terms of (5.71)

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \Big|\eta(u) - \eta(u_{n_k})\Big|\, dx\, dt \leq C_\eta \int_0^T \int_{\mathbb{R}} |u - u_{n_k}|\, dx\, dt \to 0 \\ &\implies \int_0^T \int_{\mathbb{R}} (\eta(u) - \eta(u_{n_k}))\phi_t\, dx\, dt \to 0 \end{aligned} \tag{5.74}$$

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} |q(u) - q(u_{n_k})|\, dx\, dt \leq \|u_0\|_1 C_{\eta'} \int_0^T \int_{\mathbb{R}} |u - u_{n_k}|\, dx\, dt \to 0 \\ \implies &\int_0^T \int_{\mathbb{R}} (q(u) - q(u_{n_k}))\phi_x\, dx\, dt \to 0 \end{aligned} \tag{5.75}$$

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} &|\eta'(u) D_x P^u - \eta'(u_{n_k}) D_x P^{u_{n_k}}|\, dx\, dt \\ &\leq \int_0^T \int_{\mathbb{R}} |\eta'(u) D_x P^u - \eta'(u) D_x P^{u_{n_k}}| + |\eta'(u) D_x P^{u_{n_k}} - \eta'(u_{n_k}) D_x P^{u_{n_k}}|\, dx\, dt \\ &\leq \frac{3C_{\eta'}}{4} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-|x-y|} |u(y,t)^2 - u_{n_k}(y,t)^2|\, dy\, dx\, dt \\ &\quad + \|D_x P^{u_{n_k}}\|_\infty \int_0^T \int_{\mathbb{R}} |\eta'(u) - \eta'(u_{n_k})|\, dx\, dt \\ &\leq \left(3C_{\eta'}\|u_0\|_1 + \frac{3C_{\eta'}\|u_0\|_1^2}{2}\right) \int_0^T \int_{\mathbb{R}} |u(y,t) - u_{n_k}(y,t)|\, dy\, dt \to 0 \\ &\implies \int_0^T \int_{\mathbb{R}} (\eta'(u) D_x P^u - \eta'(u_{n_k}) D_x P^{u_{n_k}})\phi\, dx\, dt \to 0 \end{aligned} \tag{5.76}$$

$$\begin{aligned} &\int_{\mathbb{R}} \Big|\eta(u(x,0)) - \eta(u_{n_k}(x,0))\Big|\, dx \leq C_\eta \|u(\cdot,0) - u_{n_k}(\cdot,0)\|_1 \to 0 \\ \implies &\int_{\mathbb{R}} \Big(\eta(u(x,0)) - \eta(u_{n_k}(x,0))\Big)\phi(x,0)\, dx \to 0 \end{aligned} \tag{5.77}$$

Using these results in the inequality (5.71), the right side of the inequality goes to zero as $n_k \to \infty$. This implies

$$\int_0^T \int_{\mathbb{R}} \eta(u)\phi_t + q(u)\phi_x - \eta'(u) D_x P^u \phi\, dx\, dt + \int_{\mathbb{R}} \eta(u(x,0))\phi(x,0)\, dx \geq 0. \tag{5.78}$$

Thus u is an entropy solution.

Combining proposition 5.5 and theorem 5.1 we deduce that u is the unique entropy solutions to the Cauchy problem (5.1) with $u_0 \in \overline{\mathscr{F}(0)}^{L^1(\mathbb{R})}$. Since all multi-shockpeakons in $\{u_n\}$ are entropy solutions satisfying

$$u_n(\cdot, 0) = u_{0,n} \to u_0, \quad u_{0,n} \in L^1 \cap BV, \ \forall n \in \mathbb{N}, \tag{5.79}$$

inequality (5.2) implies that not only a subsequence of $\{u_n\}$, but the whole sequence converges to u in $L^p(\mathbb{R} \times [0, T))$ for all $p \in [1, \infty)$. □

So far we have shown that if an initial function u_0 is an element of $\mathscr{H}$, then its corresponding unique DP entropy solution is constructible by multi-shockpeakons from $\mathscr{F}$. Next we will show that $\mathscr{H} = \overline{\mathscr{F}(0)}^{L^1(\mathbb{R})}$. This implies that a DP entropy solution u with initial function $u_0 \in L^1 \cap BV$ is $\mathscr{F}$ constructible only if $u_0 \in \mathscr{H}$.

Proposition 5.8. *The closure of $\mathscr{F}(0)$ in L^1 fulfils the following equality*

$$\overline{\mathscr{F}(0)}^{L^1(\mathbb{R})} = \mathscr{H}, \tag{5.80}$$

where,

$$\mathscr{H} = \left\{ f \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) \middle| \langle f, \phi \rangle \geq 0 \text{ and } \langle Df - f, \phi \rangle \leq 0, \ \forall \phi \in \mathcal{D}(\mathbb{R}), \text{ such that } \phi \geq 0 \right\}. \tag{5.81}$$

Proof. We know from proposition 5.5 that

$$\mathscr{H} \subseteq \overline{\mathscr{F}(0)}^{L^1(\mathbb{R})}. \tag{5.82}$$

Thus we will confirm the proposition by showing the opposite inclusion

$$\mathscr{H} \supseteq \overline{\mathscr{F}(0)}^{L^1(\mathbb{R})} \tag{5.83}$$

If $u_0 \in \overline{\mathscr{F}(0)}^{L^1(\mathbb{R})}$ then there exists, by definition, a sequence $\{u_{0,n}\}_n \subset \mathscr{F}(0)$ such that $u_{0,n}$ converges to u_0 in L^1. We will prove (5.83) by showing that u_0 satisfies these three criterions:

(i) $u_0 \in BV(\mathbb{R})$.
Pf.

For any $\varepsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that

$$\|u_{0,n}\|_1 \leq \|u_0\|_1 + \varepsilon, \quad \forall n > N_1. \tag{5.84}$$

Theorem B.2 and equation (5.10) give us that

$$\|Du_0\|(\mathbb{R}) \leq \liminf_{n \to \infty} \|Du_{0,n}\|(\mathbb{R}) \leq 3\|u_0\|_1. \tag{5.85}$$

Using definition B.1, we end up with the proposed norm

$$\|u_0\|_{BV(\mathbb{R})} = \|u_0\|_1 + \|Du_0\|(\mathbb{R}) \leq 4\|u_0\|_1. \tag{5.86}$$

(ii) $\langle u_0, \phi \rangle \geq 0$ for all non-negative $\phi \in \mathcal{D}(\mathbb{R})$.
Pf.

Since all functions in $\mathscr{F}(0)$ are non-negative, we get

$$\begin{aligned}\langle u_0, \phi \rangle &= \int_{\mathbb{R}} u_0 \phi \, dx \\ &\geq \int_{\mathbb{R}} (u_0 - u_{0,n}) \phi \, dx \\ &\to 0, \qquad \text{as } n \to \infty.\end{aligned} \tag{5.87}$$

(iii) $\langle Du_0 - u_0, \phi \rangle \leq 0$ for all non-negative $\phi \in \mathcal{D}(\mathbb{R})$.
Pf.

Observe first that

$$\begin{aligned}Du_{0,n} - u_{0,n} &= \sum_i m_i^n(0) \Big(G_i'(x - x_i^n(0)) - G_i(x - x_i^n(0)) \Big) \\ &\quad + s_i^n(0) \Big(G_i(x - x_i^n(0)) - G_i'(x - x_i^n(0) \Big) - 2 s_i^n(0) \delta_{x_i^n(0)} \\ &= \sum_i \Big(s_i^n(0) - m_i^n(0) \Big) \Big(G_i(x - x_i^n(0)) - G_i'(x - x_i^n(0)) \Big) - 2 s_i^n(0) \delta_{x_i^n(0)} \\ &\leq 0, \quad \forall x \in \mathbb{R}.\end{aligned} \tag{5.88}$$

Thus for all non-negative $\phi \in \mathcal{D}(\mathbb{R})$ we have

$$\begin{aligned}\langle Du_0 - u_0, \phi \rangle &= \int_{\mathbb{R}} (Du_0 - u_0) \phi \, dx \\ &\leq \int_{\mathbb{R}} \Big(Du_0 - u_0 - (Du_{0,n} - u_{0,n}) \Big) \phi \, dx \\ &= \int_{\mathbb{R}} (u_{0,n} - u_0)(\phi + \phi_x) \, dx \\ &\to 0, \quad \text{as } n \to \infty\end{aligned} \tag{5.89}$$

Hence $u_0 \in \overline{\mathscr{F}(0)}^{L^1(\mathbb{R})}$ implies that

$$u_0 \in \Big\{ f \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) \Big| \langle f, \phi \rangle \geq 0 \text{ and } \langle Df - f, \phi \rangle \leq 0, \ \forall \phi \in \mathcal{D}(\mathbb{R}), \text{ such that } \phi \geq 0 \Big\}. \tag{5.90}$$

Consequently $\overline{\mathscr{F}(0)}^{L^1(\mathbb{R})} \subseteq \mathscr{H}$. ☐

5.3 Related constructible solutions

If we look at the space consisting of anti-peakon and shock components

$$\mathscr{F}^- := \Big\{ u = \sum_{i=1}^n m_i G_i + s_i G_i' \Big| m_i(0) \leq -s_i(0) \leq 0 \ \text{ and } \ M = \sum_{i=1}^n m_i(0) > -\infty \Big\}, \tag{5.91}$$

we achieve, by a computation similar to (4.14), that

$$m_i(t) \leq -s_i(t) \quad \forall t \in \mathbb{R}_+. \tag{5.92}$$

And with this key property we achieve similar properties for $\mathscr{F}^-$ to those we have shown $\mathscr{F}$ has. By defining

$$H^- := \left\{ f \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) \middle| \langle f, \phi \rangle \leq 0 \text{ and } \langle Df + f, \phi \rangle \leq 0, \ \forall \phi \in \mathcal{D}(\mathbb{R}), \text{ such that } \phi \geq 0 \right\}, \tag{5.93}$$

we can state the following theorem:

Theorem 5.9. *Suppose $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then u_0's unique entropy solution to the Cauchy problem* (5.1) *is $\mathscr{F}^-$ constructible if and only if $u_0 \in \mathscr{H}^-$*

Proof. Similar to the proof of theorem 5.4, and therefore omitted. □

Chapter 6

Numerical examples

In this chapter we will use multi-shockpeakons from $\mathscr{F}$ to approximate entropy solutions of the DP equation with initial data in

$$\mathscr{H} = \left\{ f \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) \middle| \langle f, \phi \rangle \geq 0 \text{ and } \langle Df - f, \phi \rangle \leq 0,\ \forall \phi \in \mathcal{D}(\mathbb{R}),\ \text{such that } \phi \geq 0 \right\}. \tag{6.1}$$

For various initial functions in $\mathscr{H}$ we will use the algorithm of theorem 5.5 (equation (5.23)) to numerically determine the initial momentum and shock values of multi-shockpeakons u_n (for $n = 3, 6, 12, 24, 48$). Thereafter we use Matlab with the explicit Runge-Kutta solver ODE45 to solve the ODEs (3.22) for the u_n functions. Assuming u_{exact} is the exact solution, error is evaluated by taking the L^1 norm[1] of $u_n - u_{\text{exact}}$ at the following times $t = (0, 2, 5)$. However, except for $t = 0$ we do not know the exact solution for most of the initial functions we are studying. So, when the exact solution is unknown, we approximate it with a high resolution u_n function; $u_{\text{exact}} := u_{48}$ for $t > 0$, (u_{48} consists of up to 48^2 shockpeakons).

Example 6.1. *In the first example we look at the peakon function*

$$u_0(x) = 2e^{-|x|}. \tag{6.2}$$

By ODEs(3.22) *we find the DP solution to this initial function:*

$$u_{\text{exact}}(x,t) = 2e^{-|x-2t|}. \tag{6.3}$$

From the simulations we see that our approximations obtain/maintain a peakon structure just like the exact solution. But for low n-values the approximate solutions, u_n, move at slower speeds than the exact solution. This is due to the nature of the shockpeakon functions we are using to approximate the exact solution (all approximations, u_n, are initially equal to zero to the right of their rightmost shockpeakon). We see that the convergence rate is close to linear, and unfortunately, that errors grow in time. This implies the numerical method is far from perfect.

[1] We compute the L^1 norm by numerical integration of $|u_n - u_{\text{exact}}|$ restricted to $(-100, 100) \subset \mathbb{R}$. This is sufficiently correct for the initial functions we are looking at.

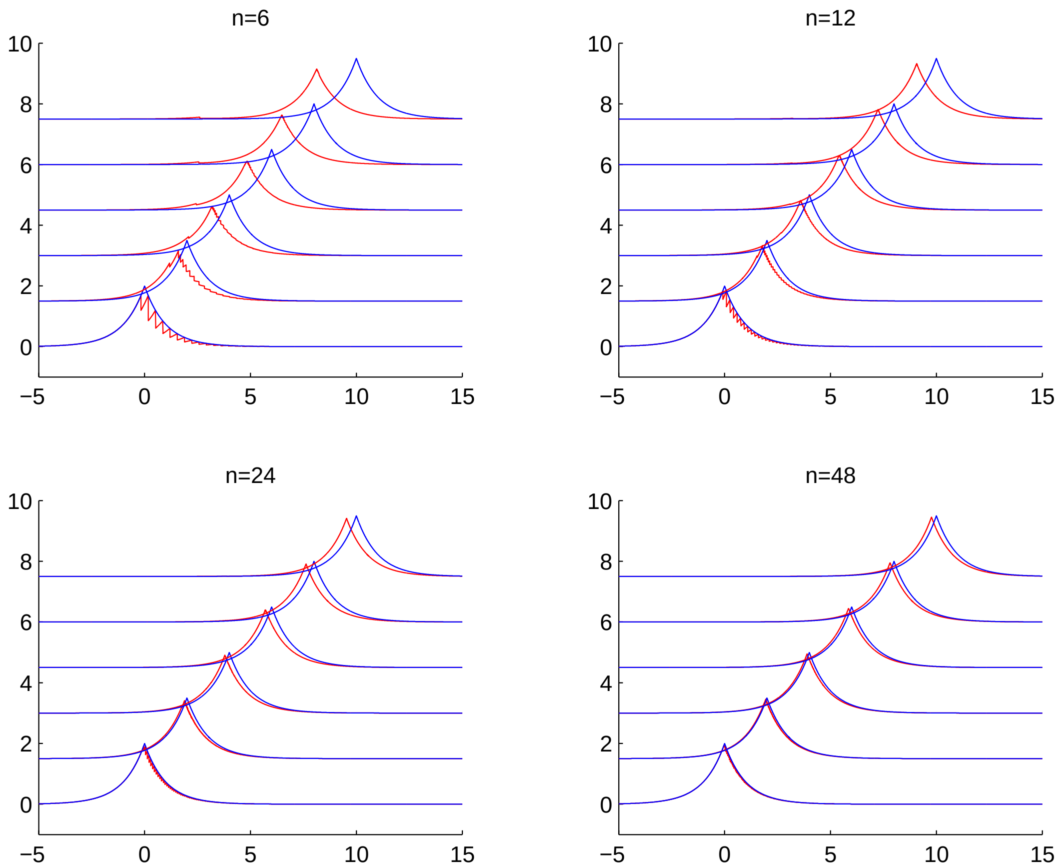

Figure 6.1: Simulation of the solution u_{exact} (in blue) and approximate solutions (in red) for the initial data (6.2) at times $t = (0, 1, 2, 3, 4, 5)$.

n-value	3	6	12	24	48
$\lVert u_n - u_{\text{exact}} \rVert_1$ at $t = 0$	1.1023	0.6337	0.3504	0.1637	0.0833
Ratio	0	1.7393	1.8087	2.1403	1.9656
$\lVert u_n - u_{\text{exact}} \rVert_1$ at $t = 2$	3.4548	2.5517	1.4555	0.7799	0.4027
Ratio	0	1.3539	1.7531	1.8663	1.9368
$\lVert u_n - u_{\text{exact}} \rVert_1$ at $t = 5$	5.3512	4.5350	2.8789	1.6256	0.8637
Ratio	0	1.1800	1.5753	1.7710	1.8822

Table 6.1: Error estimates for DP solutions of (6.2).

Example 6.2. *The second function we look at is*

$$u_0(x) = 2e^x \chi_{(-\infty,0]}(x) + 2\chi_{(0,1]}(x). \tag{6.4}$$

A multi-shockpeakon approximation to this function is illustrated in figure 5.1. The simulation in time of u_{exact} indicates that the shape of this function transforms into a peakon. But this should not be overemphasized; all our numerical simulations consist of shockpeakons, therefore it seems highly probable that they in time will transform from whatever initial shape into a set of peakons/shockpeakons. Furthermore, the error estimates are similar to those in the first example although this example uses an approximated exact solution.

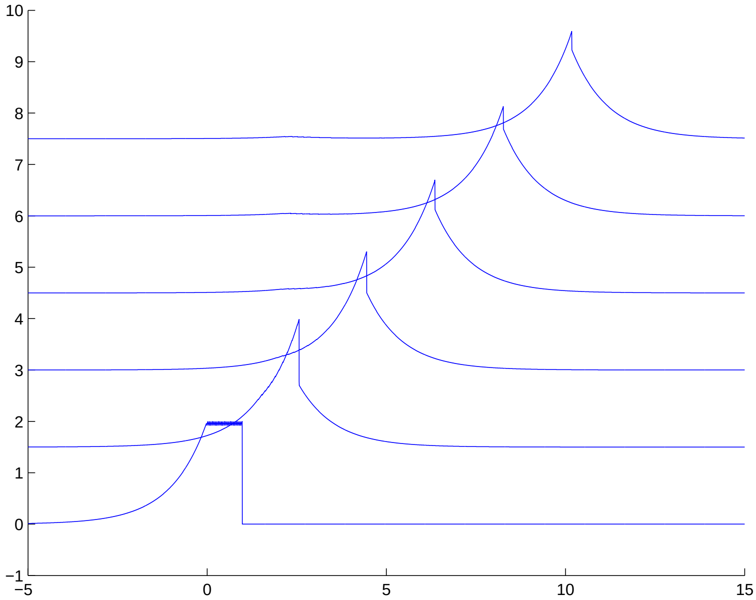

Figure 6.2: Simulation of the solution u_{exact} with initial data (6.4) at times $t = (0, 1, 2, 3, 4, 5)$.

n-value	3	6	12	24	48
$\|u_n - u_{\text{exact}}\|_1$ at $t = 0$	0.94475	0.57353	0.28181	0.15193	0.080154
Ratio		1.6473	2.0352	1.8548	
$\|u_n - u_{\text{exact}}\|_1$ at $t = 2$	3.125	2.1343	1.1324	0.42804	
Ratio		1.4642	1.8848	2.6455	
$\|u_n - u_{\text{exact}}\|_1$ at $t = 5$	4.842	3.679	2.0653	0.82184	
Ratio		1.3161	1.7813	2.513	

Table 6.2: Error estimates for DP solutions of (6.4).

Example 6.3. *Next we look at an initial function with two plateaus*

$$u_0(x) = e^{x+1}\chi_{(-\infty,-1]}(x) + \chi_{(-1,0]}(x) + (1 + e^{x-1} - e^{-1})\chi_{(0,1]}(x) + (2 - e^{-1})\chi_{(1,2]}(x). \qquad (6.5)$$

From the simulations we see that parts of the leftmost plateau disconnects from the rightmost plateau and that both parts transform into peakons.

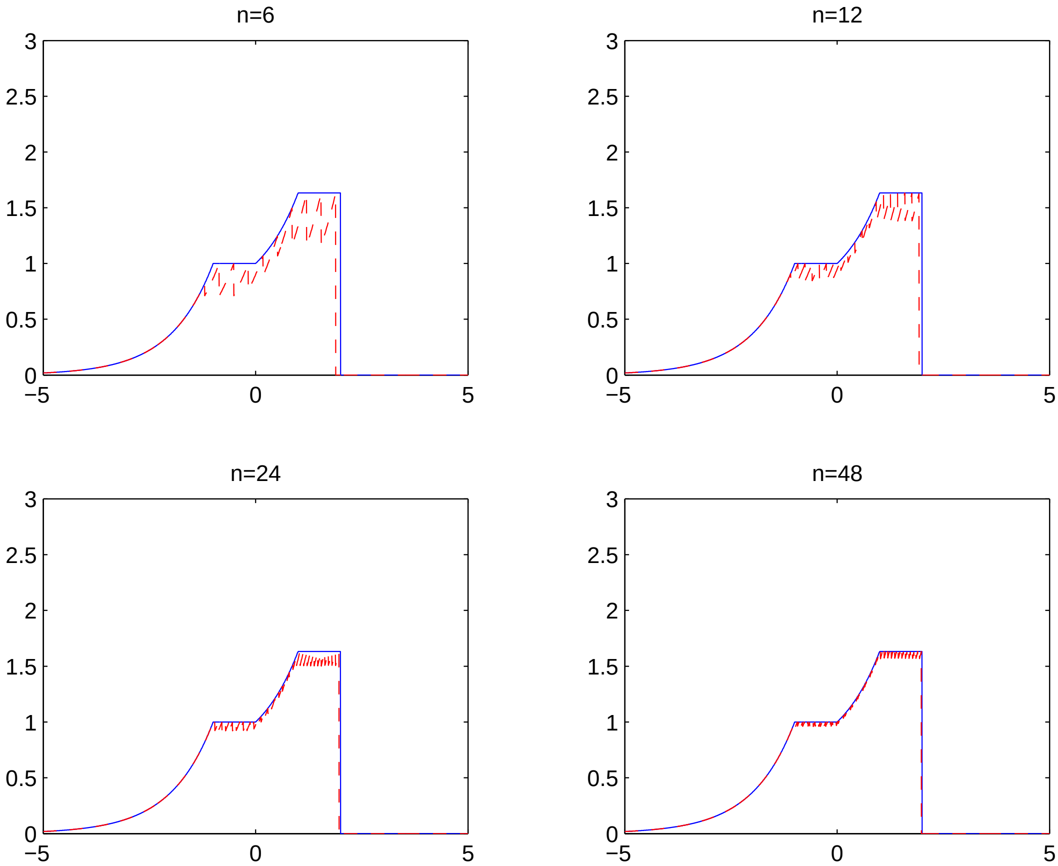

Figure 6.3: Multi-shockpeakons $\{u_n\}$ converging to equation (6.5).

n-value	3	6	12	24	48
$\lVert u_n - u_{\text{exact}} \rVert_1$ at $t = 0$	1.629	0.67827	0.36158	0.1829	0.10376
Ratio		2.4017	1.8759	1.9769	
$\lVert u_n - u_{\text{exact}} \rVert_1$ at $t = 2$	3.3686	1.8407	0.97767	0.37869	
Ratio		1.8301	1.8827	2.5817	
$\lVert u_n - u_{\text{exact}} \rVert_1$ at $t = 5$	5.3882	3.4566	1.93	0.77007	
Ratio		1.5588	1.791	2.5062	

Table 6.3: Error estimates for DP solutions of (6.5).

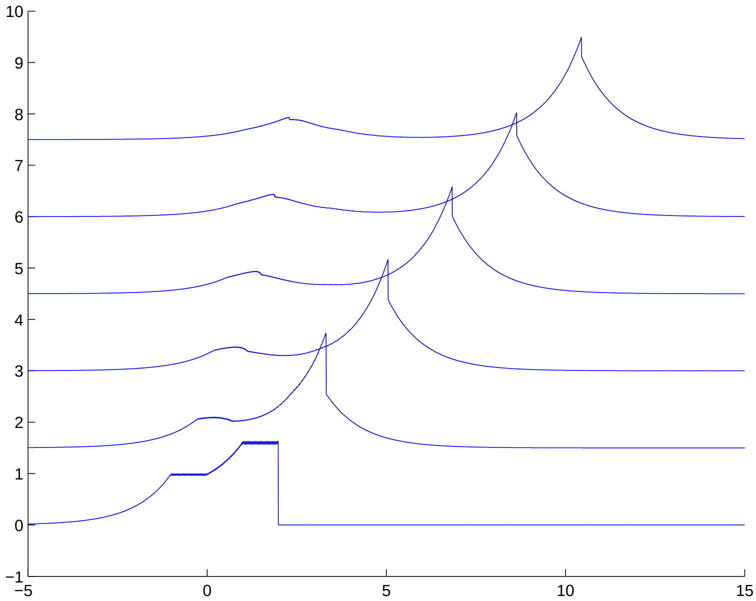

Figure 6.4: Simulation of the solution u_{exact} with initial data (6.5) at times $t = (0, 1, 2, 3, 4, 5)$.

Example 6.4. *The fourth example is the step-like function*

$$u_0(x) = 2e^{x+1}\chi_{(-\infty,-1]}(x) + 2\chi_{(-1,0]}(x) + \chi_{(0,1]}(x). \tag{6.6}$$

Because of its shape, collisions occur in this example. In time, most of the function/wave mass seems to transform into one peakon.

n-value	3	6	12	24	48
$\|u_n - u_{\text{exact}}\|_1$ at $t = 0$	1.1488	0.75019	0.3927	0.20215	0.080408
Ratio		1.5313	1.9103	1.9426	
$\|u_n - u_{\text{exact}}\|_1$ at $t = 2$	3.9329	2.8676	1.5209	0.57511	
Ratio		1.3715	1.8855	2.6445	
$\|u_n - u_{\text{exact}}\|_1$ at $t = 5$	6.3747	5.0293	2.8614	1.1391	
Ratio		1.2675	1.7576	2.5121	

Table 6.4: Error estimates for DP solutions of (6.6).

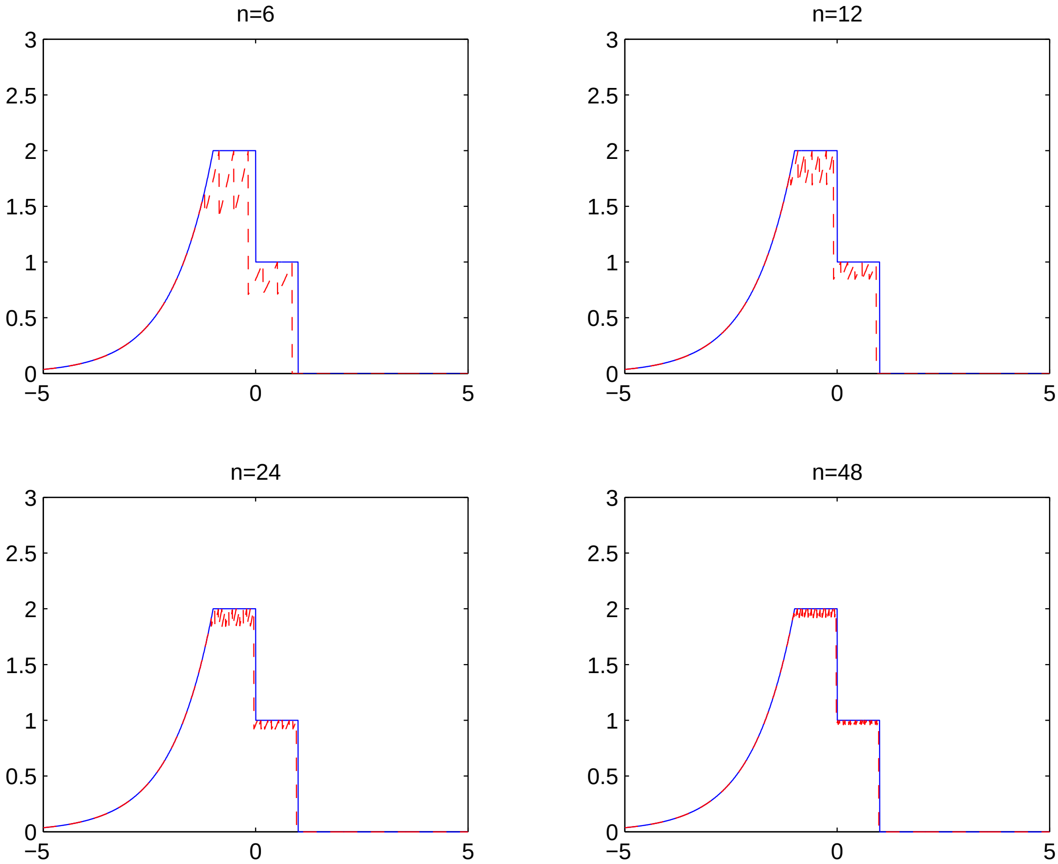

Figure 6.5: Multi-shockpeakons $\{u_n\}$ converging to equation (6.6).

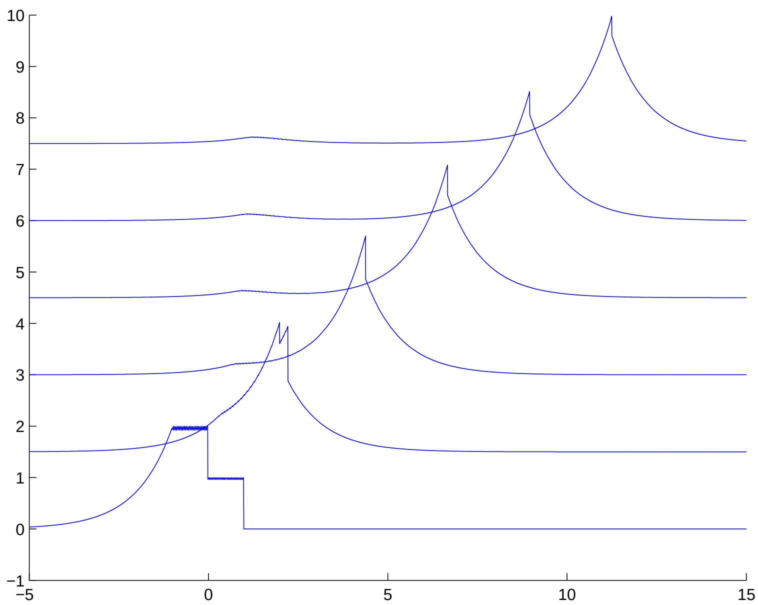

Figure 6.6: Simulation of the solution u_{exact} with initial data (6.5) at times $t = (0, 1, 2, 3, 4, 5)$.

Example 6.5. *In the last example we look at the bell shaped function*

$$u_0(x) = \frac{2}{1+x^2}. \tag{6.7}$$

As we see from the error estimates, this function is more difficult to approximate by the numerical method than the other ones we have studied. The reason is that, compared to the other functions, it is decaying very slowly as $x \to \infty$.

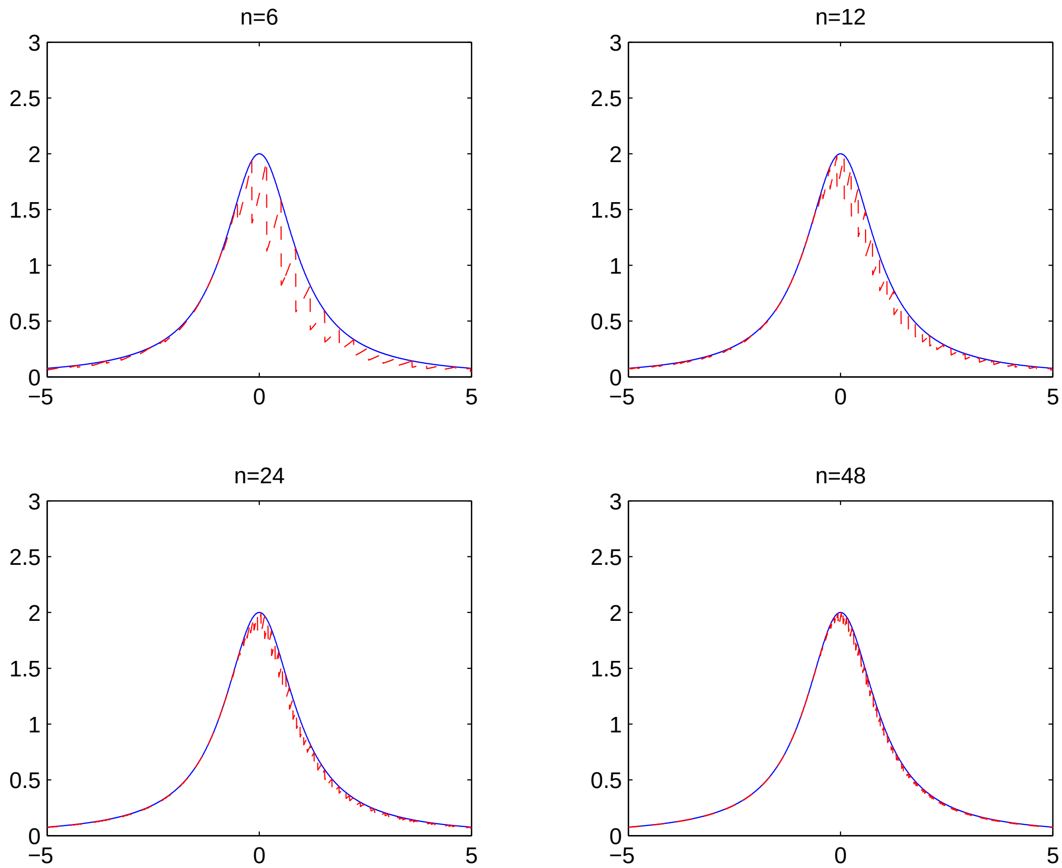

Figure 6.7: Multi-shockpeakons $\{u_n\}$ converging to equation (6.7).

n-value	3	6	12	24	48
$\|u_n - u_{\text{exact}}\|_1$ at $t = 0$	2.5341	1.4409	0.7559	0.39032	0.17123
Ratio		1.7588	1.9062	1.9366	
$\|u_n - u_{\text{exact}}\|_1$ at $t = 2$	5.4777	3.3299	1.6124	0.56651	
Ratio		1.645	2.0652	2.8462	
$\|u_n - u_{\text{exact}}\|_1$ at $t = 5$	7.466	5.6873	3.212	1.2137	
Ratio		1.3128	1.7706	2.6464	

Table 6.5: Error estimates for DP solutions of (6.7).

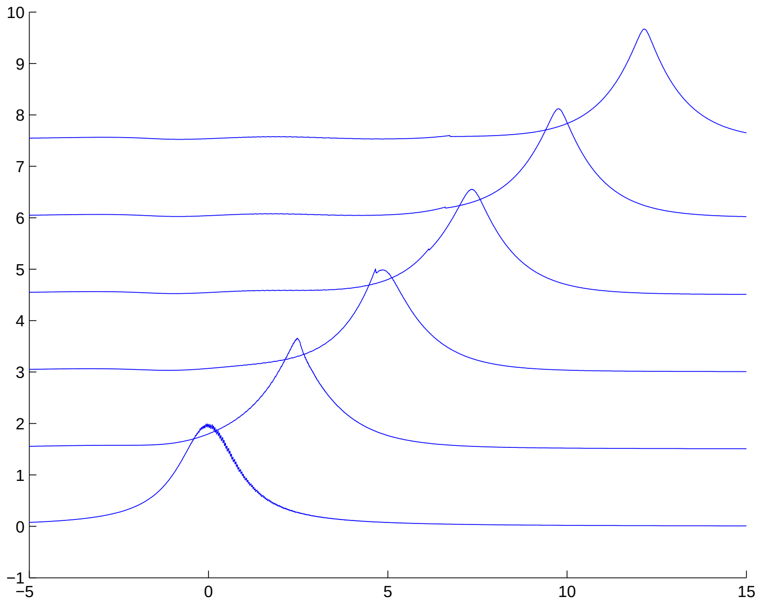

Figure 6.8: Simulation of the solution u_{exact} with initial data (6.7) at times $t = (0, 1, 2, 3, 4, 5)$.

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Appendix A

Distributional partial derivatives

Definition A.1. If $u, v \in L^1(\mathbb{R} \times \mathbb{R}_+)$ we say that $v(x,t)$ is the distributional partial derivative of $u(x,t)$ with respect to x, written $D_x u = v$, if

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty u(x,t)\phi_x(x,t)\,dxdt &= -\int_0^\infty \int_{-\infty}^\infty v(x,t)\phi(x,t),\,dxdt \\ &\text{for all test functions } \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+). \end{aligned} \tag{A.1}$$

The test functions can be nonzero at $t = 0$. This leads to the following definition of the D_t derivative.

Definition A.2. If $u, v \in L^1(\mathbb{R} \times \mathbb{R}_+)$ we say that $v(x,t)$ is the distributional partial derivative of $u(x,t)$ with respect to t, written $D_t u = v$, if

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty u(x,t)\phi_t(x,t)\,dxdt &= -\int_0^\infty \int_{-\infty}^\infty v(x,t)\phi(x,t)\,dxdt \\ &\quad -\int_{-\infty}^\infty u(x,0)\phi(x,0)\,dxdt \\ \forall \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+) \end{aligned} \tag{A.2}$$

General higher order distributional derivatives, if they exist, are found by operating step by step by the definitions above. For example, to find $D_{xt}u$ first calculate $D_x u$ then calculate $D_t(D_x u)$.

The distributional derivative of a differentiable function is its ordinary derivative.

Proposition A.3. *If* $f(x,t) \in C^1(\mathbb{R} \times \mathbb{R}_+)$, *then*

(i) $D_x f = f_x$

(ii) $D_t f = f_t$

Proof. (i)

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty D_x f(x,t)\phi(x,t)\,dxdt &= -\int_0^\infty \int_{-\infty}^\infty f(x,t)\phi_x(x,t)\,dxdt \\ &= \int_0^\infty \int_{-\infty}^\infty f_x(x,t)\phi(x,t)\,dxdt \end{aligned} \tag{A.3}$$

(ii)

$$\int_0^\infty \int_{-\infty}^\infty D_t f(x,t)\phi(x,t)\,dxdt = -\int_0^\infty \int_{-\infty}^\infty f(x,t)\phi_t(x,t)\,dxdt \\ -\int_{-\infty}^\infty f(x,0)\phi(x,0)\,dxdt \\ = \int_0^\infty \int_{-\infty}^\infty f_t(x,t)\phi(x,t)\,dxdt \tag{A.4}$$

□

The chain rule does not hold in general in distributional sense. However, for restricted class of functions it is valid.

Proposition A.4. *Functions of the form $u(x,t) = m_1(t)f(x - x_1(t))$ where $x_1, m_1 \in C^1(\mathbb{R}_+)$ satisfy the partial distributional chain rule and the partial distributional Leibniz rule. That is*

(i) $D_x(m_1(t)f(x - x_1(t))) = m_1(t)f'(x - x_1(t))$

(ii)

$$D_t(m_1(t)f(x - x_1(t)) = m_1(t)D_t f(x - x_1(t)) + f(x - x_1(t))D_t m_1(t) \\ = -m_1(t)\dot{x}_1(t)f'(x - x_1(t)) + f(x - x_1(t))\dot{m}_1(t) \tag{A.5}$$

Proof. (i)

$$\int_0^\infty \int_{-\infty}^\infty D_x(m_1(t)f(x - x_1(t)))\phi(x,t)\,dxdt \\ = -\int_0^\infty m_1(t) \int_{-\infty}^\infty f(x)\phi_x(x + x_1(t), t)\,dxdt \\ = \int_0^\infty \int_{-\infty}^\infty m_1(t)f'(x)\phi(x + x_1(t), t)\,dxdt \\ = \int_0^\infty \int_{-\infty}^\infty m_1(t)f'(x - x_1(t))\phi(x,t)\,dxdt \tag{A.6}$$

(ii) By proposition A.3 we see that $D_t m_1(t) = \dot{m}_1(t)$ and $D_t x_1(t) = \dot{x}_1(t)$. Furthermore

$$
\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty D_t\big(m_1(t) f(x - x_1(t))\big)\phi(x,t)\, dxdt \\
&\qquad = -\int_0^\infty \int_{-\infty}^\infty m_1(t) f(x)\phi_t(x + x_1(t), t)\, dxdt \\
&\qquad\quad - \int_{-\infty}^\infty m_1(0) f(x - x_1(0))\phi(x, 0)\, dx \\
&\qquad = -\int_0^\infty \int_{-\infty}^\infty m_1(t) f(x)\Big(\frac{d}{dt}\phi(x + x_1(t), t) - \dot{x}_1(t)\phi_x(x + x_1(t), t)\Big)\, dxdt \\
&\qquad\quad - \int_{-\infty}^\infty m_1(0) f(x - x_1(0))\phi(x, 0),\, dxdt \\
&\qquad = \int_0^\infty \int_{-\infty}^\infty \dot{m}_1(t) f(x)\phi(x + x_1(t), t) + \dot{x}_1(t) f(x)\phi_x(x + x_1(t), t)\, dxdt \\
&\qquad = \int_0^\infty \int_{-\infty}^\infty \big(\dot{m}_1(t) f(x) - \dot{x}_1(t) f'(x)\big)\phi(x + x_1(t), t)\, dxdt \\
&\qquad = \int_0^\infty \int_{-\infty}^\infty \big(\dot{m}_1(t) f(x - x_1(t)) - \dot{x}_1(t) f'(x - x_1(t))\big)\phi(x, t)\, dxdt
\end{aligned}
\tag{A.7}
$$

□

A special case of proposition A.4 is when $m(t) = 1$. Then we deduce that

$$
\begin{aligned}
D_x f(x - x_1(t)) &= f'(x - x_1(t)) \\
D_t f(x - x_1(t)) &= -\dot{x}_1(t) f'(x - x_1(t)).
\end{aligned}
\tag{A.8}
$$

Normalization of a function $f : \mathbb{R} \to \mathbb{R}$ means to give f the average value of left and right limits in every point x_i which is a jump discontinuity of $f; f(x_i) := \frac{f(x_i^+) + f(x_i^-)}{2}$. Normalizing a function that is smooth almost everywhere does not change its distributional derivatives, but the group of such functions has a nice property:

Proposition A.5. *Let $\mathscr{F}$ be the set of functions $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ which are smooth almost everywhere and normalized with respect to x. If $f, g \in \mathscr{F}$, then the distribution derivative D_x satisfies the Leibniz rule*

$$
D_x(fg) = f D_x g + g D_x f \tag{A.9}
$$

Proof. Except for a countable set of points $A = \{x_1, x_2, ...\}$, the function fg is smooth. At the points x_i, fg has jump discontinuities and $f(x_i) = \frac{f(x_i^+) + f(x_i^-)}{2}$, $g(x_i) = \frac{g(x_i^+) + g(x_i^-)}{2}$. We split the integral over x into two integrals; one integral over the set where the functions are smooth, $\mathbb{R} \setminus A$, and another over the set where the functions have jumps, A. From proposition A.3 we deduce that fg restricted to its smooth part, $\mathbb{R} \setminus A$, satisfy $D_x(fg) = g D_x f + f D_x g$. Defining $[f]_{x_i} := f(x_i^+) - f(x_i^-)$, we prove the theorem by a straightforward computation.

$$
\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty D_x(fg)\phi\, dxdt &= \int_0^\infty \int_{\mathbb{R}\setminus A} D_x(fg)\phi\, dxdt + \int_0^\infty \int_A D_x(fg)\phi\, dxdt \\
&= \int_0^\infty \int_{\mathbb{R}\setminus A} D_x(fg)\phi\, dxdt + \int_0^\infty \sum_{i\in A} [fg]_{x_i}\phi(x_i,t)dt \\
&= \int_0^\infty \int_{\mathbb{R}\setminus A} (gD_xf + fD_xg)\phi\, dxdt \\
&\quad + \int_0^\infty \sum_{i\in A} \Big(\frac{f(x_i^+)+f(x_i^-)}{2}[g]_{x_i} + \frac{g(x_i^+)+g(x_i^-)}{2}[f]_{x_i}\Big)\phi(x_i,t)dt \\
&= \int_0^\infty \int_{\mathbb{R}\setminus A} (gD_xf + fD_xg)\phi\, dxdt \\
&\quad + \int_0^\infty \sum_{i\in A} \Big(f(x_i)[g]_{x_i} + g(x_i)[f]_{x_i}\Big)\phi(x_i,t)dt \\
&= \int_0^\infty \int_{\mathbb{R}\setminus A} (gD_xf + fD_xg)\phi\, dxdt + \int_0^\infty \int_A (gD_xf + fD_xg)\phi\, dxdt \\
&= \int_0^\infty \int_{\mathbb{R}} (gD_xf + fD_xg)\phi\, dxdt
\end{aligned} \tag{A.10}
$$

□

A.1 Examples

In this thesis we look at the distributional derivatives of $\operatorname{sign}(x)$ and $\delta(x)$. Let us show that $D_x \operatorname{sign}(x)$ and $D_x\delta(x)$ are well defined expressions by studying their integrals.

$D_x \operatorname{sign}(x)$:

$$
\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty D_x \operatorname{sign}(x)\phi\, dxdt &= -\int_0^\infty \int_0^\infty \phi_x\, dxdt + \int_0^\infty \int_0^{-\infty} \phi_x\, dxdt \\
&= \int_0^\infty 2\phi(0,t)dt
\end{aligned} \tag{A.11}
$$

This implies that $D_x \operatorname{sign}(x) = 2\delta(x)$.

$D_x\delta(x)$:

$$
\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty D_x\delta(x)\phi\, dxdt &= -\int_0^\infty \int_{-\infty}^\infty \delta(x)\phi_x\, dxdt \\
&= \int_0^\infty -\phi_x(0,t)dt
\end{aligned} \tag{A.12}
$$

Hence δ' is a well defined test function.

Appendix B

Useful definitions and theorems

Definition B.1. A function $f \in L^1(\mathbb{R})$ has bounded variation in $\mathbb{R}$ if

$$\|Df\|(\mathbb{R}) = sup\left\{ \int_{\mathbb{R}} f\phi_x dx \middle| \phi \in C_c^1(\mathbb{R}),\ |\phi| \leq 1 \right\} < \infty. \tag{B.1}$$

This might be equivalently stated as the essential variation of $\mathbb{R}$

$$essV_{-\infty}^{\infty}(f) := sup \sum_{i}^{m} |f(y_{i+1}) - f(y_i)|, \tag{B.2}$$

where supremum is taken over all finite partitions $\{-\infty < x_1 < \ldots < x_{m+1} < \infty\}$ such that each point is a point of approximate continuity [1].

The norm is defined

$$\|f\|_{BV(\mathbb{R})} := \|f\|_{L^1(\mathbb{R})} + \|f\|_{BV(\mathbb{R})}, \tag{B.4}$$

and we write $f \in BV(\mathbb{R})$ if $\|f\|_{BV(\mathbb{R})} < \infty$.

The next theorem and its proof is taken from Evans and Gariepy [10] (thm 5.2-1)

Theorem B.2 (Lower semicontinuity of variation measure). *Suppose $f_k \in BV(\mathbb{R})$ $(k = 1, \ldots)$ and $f_k \to f$ in $L_{loc}(\mathbb{R})$. Then*

$$\|Df\|(\mathbb{R}) \leq \liminf_{k\to\infty} \|Df_k\|(\mathbb{R}). \tag{B.5}$$

Proof. Let $\phi \in C_c^1(\mathbb{R})$, $|\phi| \leq 1$,

$$\sigma_k = \begin{cases} \frac{Df_k}{|Df_k|} & \text{if } Df \neq 0, \\ 0 & \text{if } Df_k = 0 \end{cases} \tag{B.6}$$

[1] For a given function f, x is a point of approximate continuity if for each $\epsilon > 0$

$$\lim_{r\to 0} \frac{\big|[x-r, x+r] \cap \{y \big| |f(x) - f(y)| > \epsilon\}\big|}{|[x-r, x+r]|} = 0. \tag{B.3}$$

($|[a, b]|$ denotes the Lebesgue measure of $[a, b]$).

(D represents the distributional derivative). Strong convergence of f_k implies weak convergence, so

$$\begin{aligned} \int_{\mathbb{R}} f\phi_x \, dx &= \lim_{k\to\infty} \int_{\mathbb{R}} f_k \phi_x \, dx \\ &= -\lim_{k\to\infty} \int_{\mathbb{R}} \phi\sigma_k \, \|Df_k\|(dx) \\ &= -\lim_{k\to\infty} \int_{\mathbb{R}} \phi\sigma_k \, d\|Df_k\| \\ &\leq \liminf_{k\to\infty} \|Df_k\|(\mathbb{R}) \end{aligned} \tag{B.7}$$

□

Theorem B.3 (Kolmogorov's compactness theorem). *Let M be a subset of $L^P(\Omega)$, $p \in [1,\infty)$, for some open set of $\Omega \subseteq \mathbb{R}^n$. Then M is precompact if and only if the following three conditions are fulfilled:*

(i)M is bounded in $L^p(\Omega)$, i.e.

$$\sup_{u\in M} \|u\|_p < \infty \tag{B.8}$$

(ii)We have

$$\|u(\cdot+\varepsilon) - u\| \leq \lambda(|\varepsilon|) \tag{B.9}$$

for $\lambda \leq O(|\varepsilon|)$ independent of $u \in M$ (we let u equal zero outside Ω).

(iii)

$$\lim_{\alpha\to\infty} \int_{\{x\in\Omega \mid |x|\geq\alpha\}} |u(x)|^p dx = 0 \text{ uniformly for } u \in M \tag{B.10}$$

Proof. See Holden and Risebro [9] p. 296-297. □

Appendix C

Proof of theorem 5.2(vii)

A priori we have from (4.16) and (4.14) that

$$\begin{gathered} u \in \mathscr{F} \implies u = \sum_{i=1}^{n} m_i G_i + s_i G_i' \text{ where } S = \sum_{i=1}^{n} s_i \leq \sum_{i=1}^{n} m_i = M \\ |\dot{x}_i| \leq 2M, \quad |\dot{m}_i| \leq 8M^2, \quad |\dot{s}_i| \leq 2M^2, \quad \forall i \in \{1, 2, ..., n\} \end{gathered} \tag{C.1}$$

We dive right into calculations

$$\begin{aligned} &\|u(\cdot,t) - u(\cdot,w)\|_{L^1(\mathbb{R})} \\ &\qquad = \int_{\mathbb{R}} \Big| \sum_{i=1}^{n} m_i(t) G(x - x_i(t)) + s_i(t) G'(x - x_i(t)) \\ &\qquad\quad - \sum_{i=1}^{n} m_i(w) G(x - x_i(w)) + s_i(w) G'(x - x_i(w)) \Big| \, dx \\ &\qquad \leq \int_{\mathbb{R}} \sum_{i=1}^{n} \Big(\Big| m_i(t) e^{-|x - x_i(t)|} - m_i(w) e^{-|x - x_i(w)|} \Big| \\ &\qquad\quad + \Big| s_i(t) \operatorname{sign}(x - x_i(t)) e^{-|x - x_i(t)|} - s_i(w) \operatorname{sign}\left(x - x_i(w)\right) e^{-|x - x_i(w)|} \Big| \Big) dx \end{aligned} \tag{C.2}$$

Using the triangle inequality when looking at these expressions component by component, we get

$$\begin{aligned} &\int_{\mathbb{R}} |m_i(t) e^{-|x - x_i(t)|} - m_i(w) e^{-|x - x_i(w)|}| \, dx \\ &\quad \leq \int_{\mathbb{R}} |m_i(t) e^{-|x - x_i(t)|} - m_i(w) e^{-|x - x_i(t)|}| + |m_i(w) e^{-|x - x_i(t)|} - m_i(w) e^{-|x - x_i(w)|}| \, dx \\ &\quad = |m_i(t) - m_i(w)| \int_{\mathbb{R}} e^{-|x - x_i(t)|} \, dx + m_i(w) \int_{\mathbb{R}} |e^{-|x - x_i(t)|} - e^{-|x - x_i(w)|}| \, dx \\ &\quad \leq m_i(w) \Big((2e^{8M^2|t-w|} - 2) + 4(1 - e^{-\frac{|x_i(w) - x_i(t)|}{2}}) \Big) \\ &\quad \leq 2m_i(w) \Big((e^{8M^2|t-w|} - 1) + 2(1 - e^{-M|t-w|} \Big) \end{aligned} \tag{C.3}$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}} \Big| s_i(t)\,\mathrm{sign}(x - x_i(t) e^{-|x-x_i(t)|} - s_i(w)\,\mathrm{sign}(x - x_i(w)) e^{-|x-x_i(w)|} \Big|\, dx \\
&\quad \le \int_{\mathbb{R}} \Big| s_i(t)\,\mathrm{sign}(x - x_i(t)) e^{-|x-x_i(t)|} - s_i(w)\,\mathrm{sign}(x - x_i(t)) e^{-|x-x_i(t)|} \Big|\, dx \\
&\qquad + \int_{\mathbb{R}} \Big| s_i(w)\,\mathrm{sign}(x - x_i(t)) e^{-|x-x_i(t)|} - s_i(w)\,\mathrm{sign}(x - x_i(w)) e^{-|x-x_i(t)|} \Big|\, dx \\
&\qquad + \int_{\mathbb{R}} \Big| s_i(w)\,\mathrm{sign}(x - x_i(w)) e^{-|x-x_i(t)|} - s_i(w)\,\mathrm{sign}(x - x_i(w)) e^{-|x-x_i(w)|} \Big|\, dx \\
&\quad \le |s_i(t) - s_i(w)| \int_{\mathbb{R}} e^{-|x-x_i(t)|}\, dx \\
&\qquad + s_i(w) \int_{\mathbb{R}} e^{-|x-x_i(t)|} \Big| \mathrm{sign}(x - x_i(t)) - \mathrm{sign}(x - x_i(w)) \Big|\, dx \\
&\qquad + s_i(w) \int_{\mathbb{R}} \Big| e^{-|x-x_i(t)|} - e^{-|x-x_i(w)|} \Big|\, dx \\
&\quad \le s_i(w) \Big(2(e^{2M^2|t-w|} - 1) + 2(1 - e^{-|x_i(w)-x_i(t)|}) + 4(1 - e^{\frac{-|x_i(w)-x_i(t)|}{2}}) \Big) \\
&\quad \le 2 s_i(w) \Big((e^{2M^2|t-w|} - 1) + 3(1 - e^{-2M|t-w|}) \Big).
\end{aligned}
\tag{C.4}
$$

Using these results in (C.2) we end up with

$$
\begin{aligned}
\|u(\cdot,t) - u(\cdot,w)\|_{L^1(\mathbb{R})} &\le 2 \sum_{i=1}^{n} \Big[m_i(w) \Big((e^{8M^2|t-w|} - 1) + 2(1 - e^{-M|t-w|}) \Big) \\
&\qquad + s_i(w) \Big((e^{2M^2|t-w|} - 1) + 3(1 - e^{-2M|t-w|}) \Big) \Big] \\
&\le 2 \sum_{i=1}^{n} m_i(w) \Big(2(e^{8M^2|t-w|} - 1) + 5(1 - e^{-2M|t-w|}) \Big) \\
&\le 2M \Big(2(e^{8M^2|t-w|} - 1) + 5(1 - e^{-2M|t-w|}) \Big)
\end{aligned}
\tag{C.5}
$$