Mathematics and numerics for data assimilation and state estimation – Lecture 17

Summer semester 2020

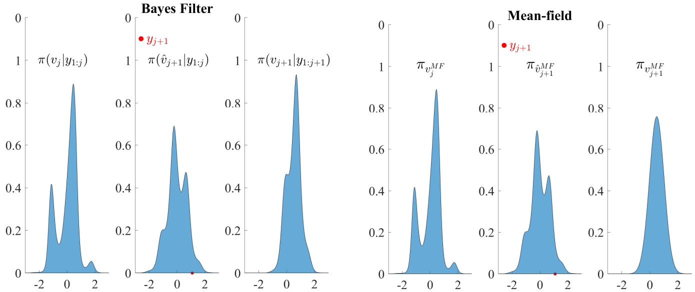
Overview

Summary lecture 16

- Described the extended KF and introduced ensemble KF.
- Studied convergence properties of the methods, particularly showing that the Gaussian approximation in the analysis of EnKF leads to errors for nonlinear problems:

$$\text{BF:} \quad \pi(v_{j+1}|y_{1:j+1}) \propto \pi_{N(y_{j+1},\Gamma)}(v_{j+1})\pi(v_{j+1}|y_{1:j})$$

$$\text{MFEnKF:} \quad \pi_{v_{j+1}^{\mathrm{MF}}}(v_{j+1}) \;\propto \pi_{K_{j+1}^{\mathrm{MF}} N(y_{j+1}-H\hat{v}_{j+1}^{\mathrm{MF}},\Gamma)} * \pi_{\hat{v}_{j+1}^{\mathrm{MF}}}(v_{j+1}).$$



Plan for today

Particle filtering: a nonlinear filtering method which, in essence, treats the prediction step as EnKF, and reweights particles in the analysis step.

Recall that for the Bayes Filter,

$$\pi(v_j|y_{1:j}) \propto \pi(y_j|v_j)\pi(v_j|y_{1:j-1}).$$

(Bootstrap) particle filters consists of collection of weights and particles: $\{(w_j^{(i)}, \hat{v}_j^{(i)})\}_{i=1}^M$ with empirical measure

$$\pi_j^M(dv) = \sum_{i=1}^M w_j^{(i)} \delta_{\hat{v}_j^{(i)}}(dv)$$

where the weights sum to 1, and

$$w_j^{(i)} \propto \pi_{Y_j|V_j}(y_j|\hat{v}_j^{(i)}),$$

and $\pi_{\hat{v}_j^{(i)}} \approx \pi_{V_j|Y_{1:j-1}}(\cdot|y_{1:j-1})$.

Overview

Filtering setting

$V_0 \sim \pi_0$ and mappings $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $G : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^k$ such that for for $j = 0, 1, \ldots$ and the hidden Markov model

$$\begin{aligned} V_{j+1} &= F(V_j, \xi_j) \\ Y_{j+1} &= G(V_{j+1}, \eta_{j+1}) \end{aligned} \tag{1}$$

with iid $\{\xi_j\}$ and iid $\{\eta_j\}$ where $V_0 \perp \{\xi_j\} \perp \{\eta_j\}$.

"Classic" setting obtained with $F(v, \xi) = \Psi(v) + \xi$ and $G(v, \eta) = h(v) + \eta$ with Gaussian ξ and η.

Note that the Markov chain $\{V_j\}$ may be associated to a time-independent kernel density function

$$\pi_{V_{j+1}|V_j}(v_{j+1}|v_j) = p(v_j, v_{j+1}).$$

and that, as in the classic setting,

$$\pi(y_{j+1}|v_{j+1}, y_{1:j}) = \pi(y_{j+1}|v_{j+1}).$$

Bayes filter – in operator notation

Notation for analysis and prediction Bayes filter pdfs:

$$\pi_j(v) := \pi_{V_j|Y_{1:j}}(v|y_{1:j}) \quad \text{and} \quad \hat{\pi}_{j+1}(v) := \pi_{V_{j+1}|Y_{1:j}}(v|y_{1:j})$$

The transition $\pi_j \mapsto \pi_{j+1}$ consists of two steps:

1. Prediction:

$$\hat{\pi}_{j+1}(v_{j+1}) = (\mathscr{P}\pi_j)(v_{j+1}) := \int_{\mathbb{R}^d} p(v_j, v_{j+1})\pi_j(v_j)dv_j$$

2. Analysis

$$\pi_{j+1}(v_{j+1}) = (\mathscr{A}_{j+1}\hat{\pi}_{j+1})(v_{j+1}) := \frac{\pi(y_{j+1}|v_{j+1})\hat{\pi}_j(v_{j+1})}{\int_{\mathbb{R}^d} \pi_{Y_{j+1}|V_{j+1}}(y_{j+1}|v)\hat{\pi}_{j+1}(v)dv}$$

   where the subscript in $\mathscr{A}_{j+1}$ relates to the value of y_{j+1}.

Summary: $\pi_{j+1} = \mathscr{A}_{j+1}\mathscr{P}\pi_j$, which may also connect to

$$\pi(v_{j+1}|y_{1:j+1}) \propto \pi(y_{j+1}|v_{j+1})\pi(v_{j+1}|y_{1:j}).$$

Bootstrap particle filter

Given a probability measure or density π, we define for any $M \in \mathbb{N}$, the empirical probability measure

$$\mathcal{S}^M \pi(dv) := \frac{1}{M} \sum_{i=1}^{M} \delta_{v^i}(dv) \quad \text{where} \quad v^{(i)} \overset{iid}{\sim} \pi.$$

Approximation ideas for particle filtering: Given π_j,

1. Approximate $\pi_j^M = \mathcal{S}^M \pi \approx \pi_j$
2. Prediction $\hat{\pi}_{j+1}^M = \mathcal{S}^M(\mathscr{P} \pi_j^M) \approx \mathscr{P} \pi_j$
3. Analysis $\pi_{j+1}^M = \mathscr{A}_{j+1} \hat{\pi}_{j+1}^M \approx \mathscr{A} \hat{\pi}_{j+1}$.

Problem: Have only defined $\mathscr{P}$ and $\mathscr{A}_{j+1}$ as mappings from pdfs to pdfs, but π_j^M and $\hat{\pi}_{j+1}^M$ a measures.

Extension of mappings

$\mathscr{P}$ as mapping from empirical probability measures (epms) to pdfs:
For any

$$\pi(dv) = \sum_{i=1}^{M} w^{(i)} \delta_{v^{(i)}}(dv)$$

with $\sum_{i=1}^{M} w^{(i)} = 1$,

$$(\mathscr{P}\pi)(u) := \int_{\mathbb{R}^d} p(v,u)\pi(dv) = \sum_{i=1}^{M} w^{(i)} p(v^{(i)}, u)$$

Example: In the classic setting $p(v,u) \propto \exp(-|\Psi(v) - u|_{\Sigma}^2/2)$ and thus

$$(\mathscr{P}\pi)(u) \propto \sum_{i=1}^{M} w^{(i)} \exp\Big(- |\Psi(v^{(i)}) - u|_{\Sigma}^2/2\Big).$$

$\mathscr{A}_j$ as mapping from epms to epms

For any epm

$$\pi(dv) = \sum_{i=1}^{M} w^{(i)} \delta_{v^{(i)}}(dv)$$

we define

$$(\mathscr{A}_j \pi)(du) := \frac{\pi_{Y_j|V_j}(y_j|u)\pi(du)}{\int_{\mathbb{R}^d} \pi_{Y_j|V_j}(y_j|v)\pi(dv)}$$

$$=$$

$$= \sum_{i=1}^{M} \frac{w^{(i)} \pi_{Y_j|V_j}(y_j|v^{(i)})}{Z} \delta_{v^{(i)}}(du)$$

with $Z = \sum_{i=1}^{M} w^{(i)} \pi_{Y_j|V_j}(y_j|v^{(i)})$.

Approximation ideas for particle filtering revisited

Given $\pi_j^M = \sum_{i=1}^M w_j^{(i)} \delta_{\hat{v}_j^i}$, we compute π_{j+1}^M by the following steps

1. Resampling $\pi_j^M = \mathcal{S}^M \pi_j^M$ $\quad \left(= \frac{1}{M} \sum_{i=1}^M \delta_{v_j^i} \right)$

2. Prediction $\hat{\pi}_{j+1}^M = \mathcal{S}^M(\mathscr{P}\pi_j^M)$ $\quad \left(= \frac{1}{M} \sum_{i=1}^M \delta_{\hat{v}_{j+1}^i} \right)$

3. Analysis

$$\pi_{j+1}^M = \mathscr{A}_{j+1} \hat{\pi}_{j+1}^M \qquad \left(= \sum_{i=1}^M \underbrace{\frac{\pi_{Y_{j+1}|V_{j+1}}(y_{j+1}|\hat{v}_{j+1}^{(i)})}{Z}}_{w_{j+1}^{(i)}} \delta_{\hat{v}_{j+1}^{(i)}} \right).$$

Note that $\pi_{j+1}^M = \mathscr{A}_{j+1} \mathcal{S}^M \mathscr{P} \pi_j^M$ is described by $\{(w_{j+1}^{(i)}, \hat{v}_{j+1}^{(i)})\}$.

Importance sampling viewpoint:

$$\pi_{j+1}(v_{j+1}) \propto \pi(y_{j+1}|v_{j+1})\pi(v_{j+1}|y_{1:j})$$
$$= \underbrace{\pi(y_{j+1}|v_{j+1})}_{\text{“weight”}} \int_{\mathbb{R}^d} \underbrace{\pi(v_{j+1}|v_j)\pi_j(v_j)}_{\text{“sampling density”}} dv_j,$$

and for the particle filters this is approximated by

$$\pi_{j+1}^M = \sum_{i=1}^M w_{j+1}^{(i)} \delta_{\hat{v}_{j+1}^{(i)}}$$

with $\hat{v}_{j+1}^{(i)} \sim \int \pi_{V_{j+1}|V_j}(\cdot|v_j)\pi_j^M(v_j)dv_j$ and $w_{j+1}^{(i)} \propto \pi_{Y_{j+1}|V_{j+1}}(y_{j+1}|\hat{v}_{j+1}^{(i)})$

Bootstrap particle filter (BPF) algorithm [SST 11.1]

- **Input:** Initial distribution π_0 (which we also write π_0^M), obs sequence $y_1, y_2, \ldots$, and M.
- **Particle generation:** For $j = 0, 1, \ldots$

  1. **Resampling** Draw $v_j^{(i)} \overset{iid}{\sim} \pi_j^M$ for $i = 1, \ldots, M$.
  2. Simulate $\hat{v}_{j+1}^{(i)} = F(v_j^{(i)}, \xi_j^{(i)})$ with iid $\xi_j^{(i)}$.
  3. Set $\bar{w}_{j+1}^{(i)} = \pi_{Y_{j+1}|V_{j+1}}(y_{j+1}|\hat{v}_{j+1}^{(i)})$
  4. and $w_{j+1}^{(i)} = \bar{w}_{j+1}^{(i)} / \sum_{k=1}^{M} \bar{w}_{j+1}^{(k)}$.
  5. Set $\pi_{j+1}^M = \sum_{i=1}^{M} w_{j+1}^{(i)} \delta_{\hat{v}_{j+1}^{(i)}}$.
- **Output:** π_j^M approximating the distribution of $V_j | Y_{1:j} = y_{1:j}$.

BPF algorithm classic setting

- **Input:** Initial distribution π_0 (which we also write π_0^M), obs sequence $y_1, y_2, \ldots$, and M.
- **Particle generation:** For $j = 0, 1, \ldots$

  1. **Resampling** Draw $v_j^{(i)} \overset{iid}{\sim} \pi_j^M$ for $i = 1, \ldots, M$.
  2. Simulate $\hat{v}_{j+1}^{(i)} = \Psi(v_j^{(i)}) + \xi_j$ with $\xi_j^{(i)} \overset{iid}{\sim} N(0, \Sigma)$.
  3. Set $\bar{w}_{j+1}^{(i)} = \exp(-\frac{1}{2}|y_{j+1} - h(\hat{v}_{j+1}^{(i)})|_\Gamma^2)$
  4. and $w_{j+1}^{(i)} = \bar{w}_{j+1}^{(i)} / \sum_{k=1}^{M} \bar{w}_{j+1}^{(k)}$.
  5. Set $\pi_{j+1}^M = \sum_{i=1}^{M} w_{j+1}^{(i)} \delta_{\hat{v}_{j+1}^{(i)}}$.
- **Output:** π_j^M.

Sequential importance sampling (SIS) vs sequential importance resampling (SIR)

- Bootstrap particle filter is a special case of SIR (can have more general "proposals" in step 2.).
- Without the resampling step 1., the particle weights multiply every step, and one may risk very uneven particle weights: this is called **the degeneracy problem**.
- With resampling, uneven weights are avoided, but (1) one may lose information and (2) the variance of the resulting particle distribution π_j^M can be shown to increase.
- Adaptive resampling can for instance be based on estimating the effective number of particles

$$n_{eff,j} \approx \frac{1}{\sum_{i=1}^{M} (w_j^{(i)})^2}$$

and employing the SIR resampling step to SIS only when $n_{eff,j} < M/10$. (Motivation: if $w_j^{(i)} = 1/M$ for all i, then $n_{eff,j} = M$.)

Sequential importance sampling algorithm 1

- **Input:** Initial distribution π_0, obs sequence $y_1, y_2, \ldots$, and M.
- **Initialization:** Draw $\hat{v}_0^{(i)} \overset{iid}{\sim} \pi_0$ and set $w_0^{(i)} = 1/M$ for $i = 1, \ldots, M$. (Hat notation here is formally "wrong" but practical.)
- **Particle and weight dynamics:** For $j = 0, 1, \ldots,$,

  1. Simulate $\hat{v}_{j+1}^{(i)} = F(\hat{v}_j^{(i)}, \xi_j^{(i)})$ with iid $\xi_j^{(i)}$.
  2. Set $\bar{w}_{j+1}^{(i)} = w_j^{(i)} \pi_{Y_{j+1}|V_{j+1}}(y_{j+1}|\hat{v}_{j+1}^{(i)})$
  3. and $w_{j+1}^{(i)} = \bar{w}_{j+1}^{(i)} / \sum_{k=1}^{M} \bar{w}_{j+1}^{(k)}$.
  4. Set $\pi_{j+1}^{M} = \sum_{i=1}^{M} w_{j+1}^{(i)} \delta_{\hat{v}_{j+1}^{(i)}}$.
- **Output:** π_j^M.

Adaptive resampling algorithm

- **Input:** Initial distribution π_0, obs sequence $y_1, y_2, \ldots$, and M.
- **Initialization:** Draw $\hat{v}_0^{(i)} \stackrel{iid}{\sim} \pi_0$ and set $w_0^{(i)} = 1/M$ for $i = 1, \ldots, M$. (Hat notation here is formally "wrong" but practical.)
- **Particle and weight dynamics:** For $j = 0, 1, \ldots,$,

1. Compute $n_{eff,j}$. If $n_{eff,j} < M/10$, then **resample:** draw $\hat{v}_j^{(i)} \stackrel{iid}{\sim} \pi_j^M$ for $i = 1, \ldots, M$ and set $w_j^{(i)} = 1/M$ for $i = 1, \ldots, M$.
2. Simulate $\hat{v}_{j+1}^{(i)} = F(\hat{v}_j^{(i)}, \xi_j^{(i)})$ with iid $\xi_j^{(i)}$.
3. Set $\bar{w}_{j+1}^{(i)} = w_j^{(i)} \pi_{Y_{j+1}|V_{j+1}}(y_{j+1}|\hat{v}_{j+1}^{(i)})$
4. and $w_{j+1}^{(i)} = \bar{w}_{j+1}^{(i)} / \sum_{k=1}^M \bar{w}_{j+1}^{(k)}$.
5. Set $\pi_{j+1}^M = \sum_{i=1}^M w_{j+1}^{(i)} \delta_{\hat{v}_{j+1}^{(i)}}$.

- **Output:** π_j^M.

Example implementation of BPF

Consider **Dynamics:**

$$\begin{aligned} V_{j+1} &= 2.5\sin(V_j) + \xi_j \\ V_0 &\sim N(0,1) \end{aligned} \tag{2}$$

where $\xi_j \sim N(0, 0.09)$ **Observations:**

$$Y_j = h(V_j) + \eta_j, \quad j = 1, 2, \dots,$$

with $\eta_j \sim N(0,1)$.

Boostrap PF:

1. Sample iid $v_0^{(i)} \sim N(0,1)$ for $i = 1, 2, \dots, M$
2. Simulate $\hat{v}_1^{(i)} = 2.5\sin(v_0^{(i)}) + \xi_0^{(i)}$ for $i = 1, 2, \dots, M$.

Bootstrap PF continued

3. Set $w_1^{(i)} \propto \exp(-\frac{1}{2}|y_1 - h(\hat{v}_1^{(i)})|_\Gamma^2)$ and normalize weights to sum to unity.

4. Set $\pi_1^M(du) = \sum_{i=1}^M w_1^{(i)} \delta_{\hat{v}_1^{(i)}}(du)$.

5. **Resampling:** Sample iid $v_1^{(i)} \sim \pi_1^M$ for $i = 1, 2, \ldots, M$

6. Simulate $\hat{v}_2^{(i)} = 2.5 \sin(v_1^{(i)}) + \xi_1^{(i)}$ for $i = 1, 2, \ldots, M$, and so forth.

How to sample from an empirical probability measure $\pi_j^M(du)$? Similar as sampling a transition in a finite state space Markov chain, cf. Lecture 5 Annotated, p. 35-36, and [SST 11.4].

Overview

Notation:

- Recall that $\mathcal{P}$ denotes the space of probability measures on $\mathbb{R}^d$, and let $\mathcal{P}_\Omega$ denote the space of **random** probability measures.
- Let now π_j denote the distribution of $V_j | Y_{1:j} = y_{1:j}$ (rather than, as before, the pdf) , and let π_j^M denote the particle filter approximation.
- For any $f : \mathbb{R}^d \to \mathbb{R}$ we define the scalar-valued rv

$$\pi_j[f] = \mathbb{E}^{\pi_j}[f] \quad \text{and} \quad \pi_j^M[f] = \mathbb{E}^{\pi_j^M}[f].$$

In order to study the large-particle-limit convergence of $\pi_j^M \to \pi_j$, we introduce the following metric on $\mathcal{P}_\Omega$ (or, equivalently, on the space of random pdfs $\mathcal{M}_\Omega$)

$$d(\pi, \tilde{\pi}) := \sup_{\|f\|_\infty \le 1} \sqrt{\mathbb{E}\left[\left(\pi[f] - \tilde{\pi}[f]\right)^2\right]},$$

for $\pi, \tilde{\pi} \in \mathcal{P}_\Omega$ (or $\in \mathcal{M}_\Omega$).

Exercise: Verify that the triangle inequality holds.

Theorem 1 (SST 11.6)

Consider the dynamics-observation setting (1), *and for a given sequence* $y_{1:J}$, *assume there exists a* $\kappa \in (0,1)$ *such that*

$$\kappa \leq \pi_{Y_j|V_j}(y_j|u) \leq \kappa^{-1} \quad \text{for all } u \in \mathbb{R}^d \quad \text{and} \quad j \in \{0,1,\ldots,J\}. \qquad (3)$$

Then, for all $j \in \{0,1,\ldots,J\}$, *it holds for the BPF algorithm that*

$$d(\pi_j, \pi_j^M) \leq \frac{c(J,\kappa)}{\sqrt{M}}.$$

Remark: The assumption (3) never holds in the classic setting! See ubung 8 for settings where an adapted assumption holds.

Sketch of proof: Recall that

$$\pi_{j+1} = \mathscr{A}_{j+1}\mathscr{P}\pi_j \quad \text{and} \quad \pi_{j+1}^M = \mathscr{A}_{j+1}\mathcal{S}^M\mathscr{P}\pi_j^M.$$

Proof of Thm 1

Hence,

$$\begin{aligned} d(\pi_{j+1}, \pi_{j+1}^M) &= d\Big(\mathscr{A}_{j+1}\mathscr{P}\pi_j,\, \mathscr{A}_{j+1}\mathcal{S}^M\mathscr{P}\pi_j^M\Big) \\ &\leq d\Big(\mathscr{A}_{j+1}\mathscr{P}\pi_j,\, \mathscr{A}_{j+1}\mathscr{P}\pi_j^M\Big) + d\Big(\mathscr{A}_{j+1}\mathscr{P}\pi_j^M,\, \mathscr{A}_{j+1}\mathcal{S}^M\mathscr{P}\pi_j^M\Big) \\ &\leq \frac{2}{\kappa^2}\Big[d\Big(\mathscr{P}\pi_j,\, \mathscr{P}\pi_j^M\Big) + d\Big(\mathscr{P}\pi_j^M,\, \mathcal{S}^M\mathscr{P}\pi_j^M\Big)\Big], \end{aligned}$$

where the last inequality used that for any $\pi, \tilde{\pi} \in \mathcal{P}_\Omega$, and $0 \leq j \leq J$,

$$d\Big(\mathscr{A}_j\pi,\, \mathscr{A}_j\tilde{\pi}\Big) \leq \frac{2}{\kappa^2} d\Big(\pi,\, \tilde{\pi}\Big). \tag{4}$$

Verification of (4): Let us write $g_j(u) := \pi_{Y_j|V_j}(y_j|u)$, and note that $\kappa \le g_j \le \kappa^{-1}$, and recall that for any $\tilde{\pi} \in \mathcal{P}$, the analysis operator is defined by

$$(\mathscr{A}_j\tilde{\pi})(du) = \frac{\pi_{Y_j|V_j}(y_j|u)\tilde{\pi}(du)}{\int \pi_{Y_j|V_j}(y_j|u)\tilde{\pi}(du)} = \frac{g_j(u)\tilde{\pi}(du)}{\tilde{\pi}[g_j]}.$$

Hence,

$$(\mathscr{A}_j\tilde{\pi})[f] = \int_{\mathbb{R}^d} f(u)(\mathscr{A}\tilde{\pi})(du) = \int_{\mathbb{R}^d} f(u)\frac{g_j(u)\tilde{\pi}(du)}{\tilde{\pi}[g]} = \frac{\tilde{\pi}[g_j f]}{\tilde{\pi}[g_j]}.$$

and

$$\begin{aligned}
|(\mathscr{A}_j\pi)[f] - (\mathscr{A}_j\tilde{\pi})[f]| &= \left|\frac{\pi[g_j f]}{\pi[g_j]} - \frac{\tilde{\pi}[g_j f]}{\tilde{\pi}[g_j]}\right| \\
&= \left|\frac{\pi[g_j f]}{\pi[g_j]} - \frac{\tilde{\pi}[g_j f]}{\pi[g_j]} + \frac{\tilde{\pi}[g_j f]}{\pi[g_j]} \quad - \quad \frac{\tilde{\pi}[g_j f]}{\tilde{\pi}[g_j]}\right| \\
&= \left|\frac{\pi[\kappa g_j f] - \tilde{\pi}[\kappa g_j f]}{\kappa\pi[g_j]} + \frac{\tilde{\pi}[g_j f]}{\tilde{\pi}[g_j]}\frac{(\tilde{\pi}[\kappa g_j] - \pi[\kappa g_j])}{\kappa\pi[g_j]}\right| \\
&\overset{\tilde{\pi}[g],\pi[g_j]>\kappa}{\le} \frac{\left|\pi[\kappa g_j f] - \tilde{\pi}[\kappa g_j f]\right|}{\kappa^2} + \left|\frac{\tilde{\pi}[g_j f]}{\tilde{\pi}[g_j]}\right|\frac{\left|\tilde{\pi}[\kappa g_j] - \pi[\kappa g_j]\right|}{\kappa^2}
\end{aligned}$$

Since

$$\left|\frac{\tilde{\pi}[g_j f]}{\tilde{\pi}[g_j]}\right| = |(\mathscr{A}_j \tilde{\pi})[f]| \leq 1,$$

when $\|f\|_\infty \leq 1$ (which we assume here), we obtain that

$$\Big((\mathscr{A}_j \pi)[f] - (\mathscr{A}_j \tilde{\pi})[f]\Big)^2 \leq \frac{2}{\kappa^4}\left(\big(\pi[\kappa g_j f] - \tilde{\pi}[\kappa g_j f]\big)^2 + \big(\tilde{\pi}[\kappa g_j] - \pi[\kappa g_j]\big)^2\right)$$

Since $g_j \leq \kappa^{-1}$, it holds that $\|\kappa g_j\|_\infty \leq 1$ and $\|\kappa g_j f\|_\infty \leq \|f\|_\infty$, it follows that

$$\begin{aligned} d(\mathscr{A}_j \pi, \mathscr{A}_j \tilde{\pi})^2 &= \sup_{\|f\|_\infty \leq 1} \mathbb{E}\left[\Big((\mathscr{A}_j \pi)[f] - (\mathscr{A}_j \tilde{\pi})[f]\Big)^2\right] \\ &\leq \sup_{\|f\|_\infty \leq 1} \frac{2}{\kappa^4}\left(\mathbb{E}\left[\big(\pi[\kappa g_j f] - \tilde{\pi}[\kappa g_j f]\big)^2 + \big(\tilde{\pi}[\kappa g_j] - \pi[\kappa g_j]\big)^2\right]\right) \\ &\leq \frac{4}{\kappa^4} \sup_{\|f\|_\infty \leq 1} \mathbb{E}\left[\Big(\pi[f] - \tilde{\pi}[f]\Big)^2\right]. \end{aligned}$$

Conclusion: $d(\mathscr{A}_j \pi, \mathscr{A}_j \tilde{\pi}) \leq \frac{2}{\kappa^2} d(\pi, \tilde{\pi})$.

We have reached

$$d(\pi_{j+1}, \pi_{j+1}^M) = \frac{2}{\kappa^2}\Big[d\Big(\mathscr{P}\pi_j,\ \mathscr{P}\pi_j^M\Big) + d\Big(\mathscr{P}\pi_j^M,\ \mathcal{S}^M\mathscr{P}\pi_j^M\Big)\Big].$$

For the last term, it follows by $\mathcal{S}^M\mathscr{P}\pi_j^M$ being an epm with iid dirac points, that

$$\begin{aligned} d\Big(\mathscr{P}\pi_j^M, \mathcal{S}^M\mathscr{P}\pi_j^M\Big) &= \sup_{\|f\|_\infty \le 1} \mathbb{E}\left[\left((\mathscr{P}\pi_j^M)[f] - \sum_{i=1}^{M} \frac{f(\hat{v}_{j+1}^{(i)})}{M}\right)^2\right] \\ &\le \sup_{\|f\|_\infty \le 1} \frac{\mathrm{Var}^{\mathscr{P}\pi_j^M}[f]}{\sqrt{M}} \le \frac{1}{\sqrt{M}}. \end{aligned}$$

And for the first term, we will show that

$$d\Big(\mathscr{P}\pi_j,\ \mathscr{P}\pi_j^M\Big) \le d\Big(\pi_j,\ \pi_j^M\Big), \tag{5}$$

Verfication of (5), for any $\pi, \tilde{\pi} \in \mathcal{P}$,

$$\begin{aligned}(\mathscr{P}\pi)[f] - (\mathscr{P}\tilde{\pi})[f] &= \int_{\mathbb{R}^d} f(v)\Big(\mathscr{P}\pi)(v) - (\mathscr{P}\tilde{\pi})(v)\Big)dv \\ &= \int_{\mathbb{R}^d} f(v) \int_{\mathbb{R}^d} p(u,v)(\pi(du) - \tilde{\pi}(du))dv \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(v)p(u,v)dv\right) (\pi(du) - \tilde{\pi}(du)) \\ &= \int_{\mathbb{R}^d} q_f(u)(\pi(du) - \tilde{\pi}(du)) = \pi[q_f] - \tilde{\pi}[q_f].\end{aligned}$$

and $\|q_f\|_\infty \leq 1$ whenever $\|f\|_\infty \leq 1$.
Consequently,

$$\begin{aligned}d\Big(\mathscr{P}\pi,\ \mathscr{P}\tilde{\pi}\Big)^2 &= \sup_{\|f\|\leq 1} \mathbb{E}\left[\Big((\mathscr{P}\pi)[f] - (\mathscr{P}\tilde{\pi})[f]\Big)^2\right] \\ &= \sup_{\|f\|\leq 1} \mathbb{E}\left[\Big(\pi[q_f] - \tilde{\pi}[q_f]\Big)^2\right] \\ &\leq \sup_{\|q\|\leq 1} \mathbb{E}\left[\Big(\pi[q] - \tilde{\pi}[q]\Big)^2\right] = d\Big(\pi,\ \tilde{\pi}\Big)^2.\end{aligned}$$

Conclusion

$$
\begin{aligned}
d(\pi_{j+1}, \pi_{j+1}^M) &= d\Big(\mathscr{A}_{j+1}\mathscr{P}\pi_j,\ \mathscr{A}_{j+1}\mathcal{S}^M\mathscr{P}\pi_j^M\Big) \\
&\le \frac{2}{\kappa^2}\Big[d\Big(\mathscr{P}\pi_j,\ \mathscr{P}\pi_j^M\Big) + d\Big(\mathscr{P}\pi_j^M,\ \mathcal{S}^M\mathscr{P}\pi_j^M\Big)\Big] \\
&\le \frac{2}{\kappa^2}\Big(d\Big(\pi_j,\ \pi_j^M\Big) + \frac{1}{\sqrt{M}}\Big) \\
&\le \dots \le \left(\frac{2}{\kappa^2}\right)^{j+1} \underbrace{d\Big(\pi_0,\ \pi_0^M\Big)}_{=0} + \frac{\sum_{k=0}^{j}\left(\frac{2}{\kappa^2}\right)^k}{\sqrt{M}}.
\end{aligned}
$$

End of proof.

Summary and next lecture

- Particle filter is an unbiased filtering method which converges weakly to the Bayes filter in the large-ensemble limit.
- It is applicable also in settings both with nonlinear Ψ and h, and also for more general hidden Markov models.
- Degeneracy is an important issue for particle filters, particularly for high-dimensional problems. It is an ongoing research topic to understand this phenomenon and develop more robust particle filters.
- Next time: Continuous time stochastic processes in the form of Wiener processes, Ito integration and Ito stochastic differential equations.