LV 11.4500 – UBUNG 2

(Exercises in boldface numbering originally appeared in Ubung 1)

Exercises from FJK. 1.3.12, 1.3.19, 2.1.8, 2.1.9, 2.1.13, 2.14

Other exercises.

U2.1 Write a computer program for simulating a simple symmetric random walk on $\mathbb{Z}$ with $X_0 = 0$.

(a) Simulate a collection of independent realizations of paths and estimate the expecated time (i.e. the value of n) it takes for the random walk to reach $|X_n| = 3$. In other words, for the rv $\bar{n} := \inf\{n \geq 0 \mid |X_n| = 3\}$, the objective is to approximate the value of $\mathbb{E}[\bar{n}]$ by a sample-average/Monte Carlo method.

Compare the result you obtain with Theorem 2.1.18 in FJK.

(b) As a "practical verification" of Donsker's theorem, simulate said random walk for $n = 10$, 100 and 1000, and plot the resulting function $W^{(n)}(t)$ for $t \in [0, 1]$ from Donsker's theorem.

U1.4 (Theorem 16, Lecture 3). Consider discrete rv $X : \Omega \to A \subset \mathbb{R}^d$ and $Y : \Omega \to B \subset \mathbb{R}^k$. Prove that if a mapping $B \ni b \mapsto g(b) \in \mathbb{R}^d$ satisfies

$$\sum_{b \in B} f(b)g(b)P(Y = b) = \sum_{a \in A, b \in B} f(b)aP(X = a, Y = b)$$

for all bounded $f : B \to \mathbb{R}$, then $g(Y) = \mathbb{E}[X|Y]$ almost surely.

Hint: Since B is countable, every bounded $f : B \to \mathbb{R}$ can be expressed as a linear combination of the basis functions $\{e_b\}_{b \in B}$ defined by

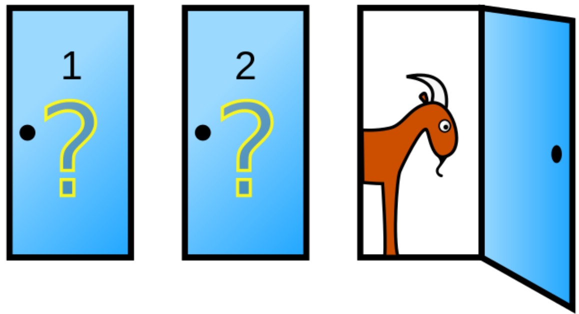
$$e_b(\bar{b}) = \begin{cases} 1 & \text{if } \bar{b} = b \\ 0 & \text{otherwise.} \end{cases}$$

U1.5 Let $X, Y \sim Bernoulli(1/2)$ be independent and define

$$Z := \begin{cases} 0 & \text{if } X + Y \in \{0, 2\} \\ 1 & \text{if } X + Y = 1 \end{cases}$$

Prove that $X \perp Z$ and $Y \perp Z$ but that that the triple $\{X, Y, Z\}$ is not independent.

Conclusion: For a sequence of rv $\{X_i\}$, pairwise independence does not imply independence!



U1.6 The Monty Hall problem (from Wikipedia): “Suppose you’re on a game show, and you’re given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what’s behind the doors, opens another door, say No. 3, which has a goat. He then says to you, ”Do you want to pick door No. 2?” Is it to your advantage to switch your choice?”

Under the classic assumptions (see Wikipedia) it is well known (and perhaps a bit surprising?) that your probability of winning increases from $1/3$ to $2/3$ by switching doors.

In this exercise we will change the classic assumptions of the Monty Hall problem to investigate whether the switching strategy generally is to be preferred over the alternative. Consider the following assumptions:

- Let $C : \Omega \to \{1, 2, 3\}$ denote location-of-the-car rv, with $\{C = k\}$ denoting the event that the car is behind door k. Assume that $\mathbb{P}(\{C = k\}) = 1/3$ for $k = 1, 2, 3$.
- Let $D : \Omega \to \{1, 2, 3\}$ denote your initial door-pick rv, which we assume has uniform probability

$$\mathbb{P}(\{D = k\}) = 1/3 \quad \text{for} \quad k = 1, 2, 3,$$

  and that $C \perp D$.
- **Assume the host will only occasionally open a door containing a goat and offer you to switch doors among the remaining unopened ones.** When $\{C \neq D\}$ then the host opens a door and offers switch with probability $1/3$, while when the complement event $\{C = D\}$ occurs, then the host always opens a door and offers switch. Whenever the host opens a door, he always opens the one among the available goat-hiding doors with the highest number (this is irrelevant for this exercise, but just to be clear). To describe this in terms of an rv, let $S : \Omega \to \{0, 1\}$ denote the host’s door-opens-and-switch-is-offered rv, with $\{S = 0\}$ denoting the event that door-opens-switch-is-offered does not occur and $\{S = 1\}$ that it does occur. By the above, $\mathbb{P}(S = 1 \mid C = j, D = k) = 1/3$ for any $j, k \in \{1, 2, 3\}$ with $j \neq k$ and $\mathbb{P}(S = 1 \mid C = D) = 1$.

Your task is to compare two very simple strategies: either always switching doors when the host offers you the possibility or never switching doors.

Let D^* denote the door-pick random value for the always-switch-when-offered strategy.

a) Write a computer program which draw random samples of C, D and D^* a large number of times and estimate in a Monte Carlo/sample average manner the probability of winning the car if always switching doors and if never switching doors.

b) Verify mathematically that the probability of winning the car when pursuing the no-switch strategy is $1/3$

c) Show that

$$\mathbb{P}(D^* = 1, C = 1) = \sum_{k=2}^{3} \mathbb{P}(D^* = 1 | C = 1, S = 1, D = k)\mathbb{P}(S = 1 | C = 1, D = k)\mathbb{P}(C = 1, D = k)$$

Hint: By the law of total probability

$$\mathbb{P}(D^* = 1, C = 1) = \mathbb{P}(D^* = 1, C = 1, S = 0) + \mathbb{P}(D^* = 1, C = 1, S = 1)$$

and use conditional probabilities.

d) Compute the probability of winning the car when pursuing the always-switch-when-offered strategy.

e) All other assumptions left as is, what is the threshold probability-value for $\mathbb{P}(S = 1 \mid C = j, D = k)$ (assuming it takes the same value for all $j, k = \{1, 2, 3\}$ with $j \neq k$), such that both strategies are equally likely to be successful?