LV 11.4500 – UBUNG 4

U4.1 a)Let $X, Y \in L^2(\Omega)$ be scalar-valued rv. Show that if $X \perp Y$, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$$

b) Let $(X, Y) \sim U(0, 1)^2$. Verify that $X \perp Y$.

c) Let $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ be scalar-valued rv. Show that if (X, Y) is multivariate normal and

$$\mathrm{Cov}[X, Y] = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = 0,$$

then $X \perp Y$. Hint: joint pdf.

d) Let $X \sim U(-1, 1)$ and $Y \sim U(-1, 1)$ with $X \perp Y$ and $Z = X + Y$. Compute the pdf $\pi_Z(z)$.

U4.2 Let $(X, Y) \sim N\left(\begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix}, \begin{bmatrix}1 & \rho \\ \rho & 1\end{bmatrix}\right)$ with $\rho \in (-1, 1)$. Compute $\pi_{X|Y}(x|y)$.

U4.3 Let X, Y and Z be rv defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and assume that $X \in L^2(\Omega)$. Show that

(i) $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$,

(ii) $\mathbb{E}[X|\mathcal{V}] = X$ a.s., for any sigma-algebra $\mathcal{V}$ satisfying $\sigma(X) \subset \mathcal{V} \subset \mathcal{F}$,

(iii) if $\sigma(Z) \subset \sigma(Y) \subset \mathcal{F}$ then

$$\mathbb{E}[\mathbb{E}[X|Y]\,|Z] = \mathbb{E}[\mathbb{E}[X|Z]\,|Y] = \mathbb{E}[X|Z] \quad \text{a.s.}.$$

U4.4 a) Verify that the Hellinger distance is a metric.

b) Consider an Bayesian inverse problem $Y = G(U)+\eta$ with prior density with compact support: For

$$A = \{u \in \mathbb{R}^d \mid \pi_U(u) > 0\},$$

it holds that $\max\{|u| \mid u \in A\} \leq 1$. Assume further that $\eta \sim N(0, 1)$, an observation $Y = y$ is given and that G_δ is a perturbation of the forward model G giving rise to a perturbed posterior density $\pi^\delta(u|y)$. State constraints on mappings $f : \mathbb{R}^d \to \mathbb{R}$ ensuring that

$$|\mathbb{E}^{\pi(\cdot|y)}[f] - \mathbb{E}^{\pi^\delta(\cdot|y)}[f]| \leq d_{TV}(\pi(\cdot|y), \pi^\delta(\cdot|y)),$$

and state constaints on G_δ such that

$$d_{TV}(\pi(\cdot|y), \pi^\delta(\cdot|y)) \leq C\delta$$

for some $C > 0$.

c) Consider the Bayesian inverse problem

$$Y = U + \eta$$

with $U, \eta \sim U(0,1)$ and $U \perp \eta$. For $Y = 0$, where we assume that $\pi_Y(y) > 0$, compute the posterior $\pi_{U|Y}(u|0)$. Explain why this fails in producing a posterior density.

Hint: Check the consistency of the assumptions.

U4.5 Consider the fair coin example in Lecture 10. Let $y = (y_1, y_2, \ldots)$ be a sequence with cumulative sums $\bar{y}_n = \sum_{k=1}^{n} y_k$ given by

$$\bar{y}_{10} = 4, \quad \bar{y}_{100} = 48, \quad \bar{y}_{1000} = 532, \quad \bar{y}_{10000} = 5267$$

Explore numerically the probability that U is a fair coin given these measurements.

Hint: implemented naively, you may not be able to normalize your density properly.

U4.6 a) State sufficient conditions for a density π to ensure that its posterior mean equals its maximum posterior, i.e.,

$$u_{PM}[\pi] = u_{MAP}[\pi]. \tag{1}$$

b) Verify that (1) holds for any Gaussian pdf.