# LV 11.4500 – UBUNG 6

U6.1 An rv $X$ on $\mathbb{R}^d$ can be uniquely described by its distribution $\mathbb{P}_X$, its cdf $F_X$, or, when it exists, its pdf $\pi_X$. The characteristic function $\varphi : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\varphi(t) = \mathbb{E}\left[\exp(i\langle t, X\rangle)\right] \quad \forall t \in \mathbb{R}^d$$

where $i = \sqrt{-1}$ and $\langle x, y\rangle := \sum_{i=1}^{d} x_i y_i$, is yet another way to uniquely describe the rv $X$.

The characteristic function of a Gaussian rv $X \sim N(m, C)$ is given by

$$\varphi(t) = \exp(i\langle t, m\rangle - \langle t, Ct\rangle/2).$$

a) Show that if $X \sim N(m_1, C_1)$ and $Y \sim N(m_2, C_2)$ are independent rv on $\mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$, then $Z = X + AY$ is also Gaussian. Determine the mean and covariance of $Z$.

b) Consider the following dynamics

$$\begin{aligned} V_{j+1} &= AV_j + \xi_j \\ V_0 &\sim N(m_0, C_0) \end{aligned}$$

with $A \in \mathbb{R}^{d\times d}$, an iid sequence $\xi_j \sim N(0, \Sigma)$ and $V_0 \perp \{\xi_j\}$. Show that for any $j \in \mathbb{N}$, $V_j$ is Gaussian, and determine its mean and covariance.

**Hint:** Argue by induction and use a) to verify Gaussianity.

c) Consider the scalar-valued special case of the above dynamics

$$\begin{aligned} V_{j+1} &= \lambda V_j + \xi_j \\ V_0 &\sim N(m_0, \sigma_0^2) \end{aligned}$$

with an iid sequence $\xi_j \sim N(0, \sigma^2)$ that is independent from $V_0$, and a scalar-valued $\lambda$ satisfying $|\lambda| < 1$.

Verify that

$$\mathbb{P}_{V_n} \Rightarrow N(0, \frac{\sigma^2}{1-\lambda^2})$$

**Hint:** First verify that $\{\mathbb{E}\left[V_j\right]\}_j$ and $\{\mathbb{E}\left[V_j^2\right]\}_j$ are Cauchy sequences.

U6.2 In its original form (but still keeping our parameter lettering $a, b, r$ rather than the original $\sigma, \beta, \rho$) the Lorenz '63 model is given by

$$\begin{aligned} \dot{v}_1 &= a(v_2 - v_1) \\ \dot{v}_2 &= rv_1 - v_2 - v_1 v_3 \qquad t \geq 0, \\ \dot{v}_3 &= v_1 v_2 - bv_3 \end{aligned}$$

with $a, b, r > 0$ and $v(0) \in \mathbb{R}^3$.

a) Show that by a change of variables the system can be rewritten

$$\left.\begin{aligned} \dot{v}_1 &= a(v_2 - v_1) \\ \dot{v}_2 &= -av_1 - v_2 - v_1 v_3 \\ \dot{v}_3 &= v_1 v_2 - b v_3 - b(r+a) \end{aligned}\right\} =: f(v), \qquad t \geq 0, \tag{1}$$

with $a, b, r$ as above, and $v(0) \in \mathbb{R}^3$.

b) Determine $\alpha, \beta > 0$ (as functions of $a, b, r$) such that

$$f(v)^T v \leq \alpha - \beta |v|^2.$$

c) Assuming a solution $v$ exists for all times $t \geq 0$ with $|v(0)|^2 \leq \alpha/\beta$, show that

$$|v(t)|^2 \leq \alpha/\beta \quad \forall t \geq 0.$$

d) Assuming that $|v(0)|^2 \leq \alpha/\beta$, prove that there exists a locally unique solution up until the first time when $|v(t)|^2 > \alpha/\beta$. Conclude from c) that a unique solution exists for all times.

**Hint:** Apply the Picard-Lindelöf theorem.

Let $D = \{v \in \mathbb{R}^3 | |v|^2 \leq \alpha/\beta\}$. Then the restriction of the above vector field $f : D \to \mathbb{R}^3$ is uniformly Lipschitz continuous on $D$.

e) As a very simplified numerical study of how partial observations may improve the stability of ODE, consider the Lorenz '63 dynamics (1) with $(a, b, r) = (10, 8/3, 28)$ with either $v(0) = (1, 1, 1)$ or perturbed initial data $\tilde{v}(0) = (1, 1, 1 + 10^{-5})$. To study the stability of the dynamics, we consider $|v(20) - \tilde{v}(20)|$ solved by numerical integration in the following Matlab implementation:

```
options = odeset('RelTol',1e-12,'AbsTol',1e-10);
a = 10;
b = 8/3;
r = 28;
f = @(t,v) [a*(v(2)-v(1));
-a*v(1)-v(2)-v(1)*v(3);
v(1)*v(2)-b*v(3)-b*(r+a)];
[t,v]=ode45(f,[0 20],[1 1 1], options);
[t2,vTilde] = ode45(f,[0 20],[1 1 1+1e-5], options);
finalTimeError = norm(v(end,:) - vTilde(end,:))
```

This yields $|v(20) - \tilde{v}(20)| \approx 15.7$.

As a comparison, try to update the value of the third component of the dynamics $\tilde{v}$ at every integer time $t = 1, 2, \ldots, 19$ with an exact observation of $v_3$. That is, set $v(0) = (1, 1, 1)$ and $\tilde{v}(0) = (1, 1, 1 + 10^{-5})$ and for $t = 0, 1, \ldots, 19$:

1. compute $v(t+1) = \Psi(v(t); 1)$ and $\tilde{v}(t+1) = \Psi(\tilde{v}(t); 1)$
2. update/correct third component of the perturbed dynamics with exact observation, $\tilde{v}_3(t+1) = v_3(t+1)$
3. set $t \mapsto t + 1$ and return to step 1.

Here, $\Psi(\cdot;\tau)$ is defined as on page 13 of Lecture 13.

Implement this algorithm in your favorite computer language and compute the resulting final time error $|v(20) - \tilde{v}(20)| = ?$.

Rather than updating the value of the third component of $\tilde{v}_3$, try updating any of the other components and see whether that improves the stability to the same degree.

U6.3 Consider the $V_0|Y_{1:J} = y_{1:J}$ smoothing problem on page 35 of Lecture 13 with $|\lambda| = 1$. Verify that $V_0|Y_{1:J} = y_{1:J} \sim N(m(j), \sigma^2_{post}(j))$ and show that

$$\sigma^2_{post}(J) \to 0 \quad \text{as} \quad J \to \infty.$$

Interpret the result.

U6.4 The derivation of smoothing densities treated in the lectures considers dynamics additive Gaussian noise: $V_{j+1} = \Psi(V_j) + \xi_j$ with $\xi \sim N(0, \Sigma)$. This may be extended to more general Markov chain dynamics:

a) Assume that $V_0 \sim \pi_{V_0}$ and that $V_j$ is a time-homogeneous Markov chain described in terms of the transition kernel density $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$, and that we have observations

$$Y_j = h(V_j) + \eta_j, \quad j = 1, 2, \ldots$$

with iid $\eta_j \sim N(0, \Gamma)$ and $\{\eta_j\} \perp \{V_j\}$. Using the kernel density, derive, up to a constant, an expression for the smoothing posterior

$$\pi(v_{0:J}|y_{1:J}).$$

b) Consider $d = 1$ with the kernel density $k(x, y) = e^{-2|x-y|}$, $\pi_{V_0}(x) = e^{-2|x|}$, and let the observations be given by

$$Y_j = h(V_j) + \eta_j, \quad j = 1, 2, \ldots$$

with iid $\eta_j$ where $\pi_\eta(x) = e^{-2|x|}$ and $\{\eta_j\} \perp \{V_j\}$. Derive up to a constant, an expression for the smoothing posterior

$$\pi(v_{0:J}|y_{1:J}).$$

and show that if $\tilde{y}_{1:J}$ is a perturbed observation sequence of $y_{1:J}$, then there exists a $c > 0$ depending on $y_{1:J}$ and $\tilde{y}_{1:J}$ such that

$$d_H(\pi(\cdot|y_{1:J}), \pi(\cdot|\tilde{y}_{1:J})) \leq c \sum_{j=1}^{J} |y_j - \tilde{y}_j|.$$

U6.5 Consider the dynamics

$$\begin{aligned} V_{j+1} &= \sin(V_j)V_j + \xi_j \qquad j = 0, 1, \ldots \\ V_0 &\sim N(0, 1) \end{aligned}$$

with iid $\xi_j \sim N(0, 1/4)$ and observations

$$Y_j = V_j + \eta_j, \quad j = 1, 2, \ldots$$

with iid $\eta_j \sim N(0, 0.04)$ and $V_0 \perp \{\xi_j\} \perp \{\eta_j\}$. For any $J > 0$, we define the distance (functional) of a path $v_{0:J} \in \mathbb{R}^{J+1}$ by

$$D(v_{0:J}) = \sqrt{\sum_{k=0}^{J-1} |v_{k+1} - v_k|^2}$$

Given the observation

$$y_{1:10} = (0.2781, 0.8839, 1.1496, 0.6607, 0.1846, -0.5131, 0.0733, 1.3827, 0.8426, 0.4538)$$

the task of this exercise is to provide a numerical estimate of the average distance of a path $V_{0:10}$ given $Y_{1:10} = y_{1:10}$. In other words, to approximate

$$\mathbb{E}\left[\, D(V_{0:10}) | Y_{1:10} = y_{1:10}\right].$$

**Hint:** First derive the posterior $\pi(v_{0:10}|y_{1:10})$ up to a constant. Then sample the posterior by e.g. Markov Chain Monte Carlo.