LV 11.4500 – UBUNG 8

U8.1 In this exercise we will study theoretical properties of the EnKF method.

Consider the linear-Gaussian filter problem with $V_0 \sim N(m_0, C_0)$ and

$$V_{j+1} = AV_j + \xi_j, \qquad \xi_j \overset{iid}{\sim} N(0, \Sigma),$$
$$Y_{j+1} = HV_{j+1} + \eta_{j+1}, \qquad \eta_{j+1} \overset{iid}{\sim} N(0, \Gamma),$$

with $\Sigma, \Gamma, C_0 > 0$ and $\{V_0\} \perp \{\xi_j\} \perp \{\eta_j\}$ and $H \in \mathbb{R}^{k \times d} \setminus \{0\}$.

a) Let $M = \infty$ and consider the iid MFEnKF prediction ensemble at time $j+1$: $\{\hat{v}_{j+1}^{MF,(i)}\}_{i=1}^{M}$. Recall that

$$\hat{m}_{j+1} = \mathbb{E}\left[\hat{v}_{j+1}^{MF,(\cdot)}\right], \quad \text{and} \quad \hat{C}_{j+1} = \mathrm{Cov}[\hat{v}_{j+1}^{MF,(\cdot)}]$$

and assume that

$$\hat{m}_{j+1} = \hat{m}_{j+1}^{KF}, \quad \text{and} \quad \hat{C}_{j+1} = \hat{C}_{j+1}^{KF},$$

where $(\hat{m}_{j+1}^{KF}, \hat{C}_{j+1}^{KF})$ denotes the reference Kalman filter mean and covariance moments.

Having computed the prediction covariance (we suppress particle notation as they are all identically distributed)

$$\hat{C}_{j+1} = \mathrm{Cov}[\hat{v}_{j+1}^{MF}]$$

and

$$K_{j+1} = \hat{C}_{j+1} H^T (H \hat{C}_{j+1} H^T + \Gamma)^{-1}$$

we consider two different analysis approaches in MFEnKF: 1.perturbed observations (the one we have presented for EnKF in the lectures):

$$y_{j+1}^{(i)} = y_{j+1} + \eta_{j+1}^{(i)}, \qquad \eta_{j+1}^{(i)} \overset{iid}{\sim} N(0, \Gamma)$$
$$v_{j+1}^{MF,(i)} = (I - K_{j+1}H)\hat{v}_{j+1}^{MF,(i)} + K_{j+1} y_{j+1}^{(i)}$$

and 2. unperturbed observations:

$$\tilde{v}_{j+1}^{MF,(i)} = (I - K_{j+1}H)\hat{v}_{j+1}^{MF,(i)} + K_{j+1} y_{j+1}$$

Task: Show that

$$\mathrm{Cov}[\tilde{v}_{j+1}^{MF}] \neq \mathrm{Cov}[v_{j+1}^{MF}] = C_{j+1}^{KF}$$

Hint: Use that K_{j+1} is deterministic.

Remark: This is a motivation for introducing perturbed observations for EnKF (i.e., also in the non-mean-field setting of $M < \infty$).

b) Show that for the EnKF ensemble $\{\hat{v}_j^{(i)}\}_{i=1}^{M}$, it holds for any i that $v_j^{(i)} \in \mathrm{Span}(\{\hat{v}_j^{(i)}\}_{i=1}^{M})$.

c) Consider the above problem with $\Sigma = 0$. Show that for any i and $j > 0$, $v_j^{(i)} \in \mathrm{Span}(A_1, A_2, \dots, A_d)$ with A_k denoting the k-th column of A.

U8.2 Verify that for the space random probability measures on $\mathbb{R}^d$ denoted by $\mathcal{P}_\Omega$,

$$d(\pi, \tilde{\pi}) := \sup_{\|f\|_\infty \le 1} \sqrt{\mathbb{E}\left[(\pi[f] - \tilde{\pi}[f])^2 \right]}$$

is a metric.

U8.3 Consider the HMM filtering problem of similar to that in Lecture 17: $V_0 \sim \pi_0$, a mapping $F : \mathbb{R}^d \times \mathbb{R}^d \to D \subset \mathbb{R}^d$ and for $j = 0, 1, \dots$

$$\begin{aligned} V_{j+1} &= F(V_j, \xi_j) \\ Y_{j+1} &= V_{j+1} + \eta_{j+1} \end{aligned} \tag{1}$$

with iid rv $\{\xi_j\}$, iid $\eta_j \sim N(0, \Gamma)$, $\Gamma > 0$, and $V_0 \perp \{\xi_j\} \perp \{\eta_j\}$. Assume that D is a compact and that $\mathbb{P}(V_0 \in D) = 1$. Given $y_{1:J}$, find an explicit κ such that assumption (2) of the following changed version of Theorem 1, Lecture 17 holds. Furthermore, provide a short argument on how the proof of the theorem in the lecture needs to be updated for the here stated theorem to hold.

Theorem For the dynamics-observation setting (1), with a given sequence $y_{1:J}$, assume there exists a $\kappa \in (0, 1)$ such that

$$\kappa \le \pi_{Y_j|V_j}(y_j|u) \le \kappa^{-1} \quad \text{for all } j \in \{0, 1, \dots, J\} \quad \text{and} \quad u \in D. \tag{2}$$

Then, for all $j \in \{0, 1, \dots, J\}$, it holds for the BPF algorithm that

$$d(\pi_j, \pi_j^M) \le \frac{c(J, \kappa)}{\sqrt{M}}.$$

End Theorem.

U8.4 Consider the linear-Gaussian filtering problem

$$V_{j+1} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} V_j + \xi_j,$$

$$V_0 \sim N\Big(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix} \Big)$$

where $\xi_j \overset{iid}{\sim} N(0, \Sigma)$ with $\Sigma = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.1 \end{bmatrix}$.

And **observations** on $\mathbb{R}$:

$$Y_j = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{H} V_j + \eta_j, \qquad \eta_j \overset{iid}{\sim} N(0, 1/4).$$

a) Generate an observation sequence $y_{1:100}$ from synthetic data: $y_j = v_j^\dagger + \eta_j$ and compute the resulting reference analysis moments (m_j^{KF}, C_j^{KF}) by Kalman filtering.

b) Solve the filtering problem by the EnKF method for different values of the ensemble size M. Measure the performance in terms of

$$\frac{1}{J+1}\sum_{j=0}^{J}\mathbb{E}\left[|E_M[v_j^{(\cdot)}]-m_j^{KF}|^2\right]$$

for the above observation sequence (with $J = 100$) and study the convergence rate.

c)Solve the filtering problem by the SIS particle filtering for different values of the ensemble size M. Again, measure the performance in terms of

$$\frac{1}{J+1}\sum_{j=1}^{J}\mathbb{E}\left[|E_M[v_j^{(\cdot)}]-m_j^{KF}|^2\right]$$

and study the convergence rate. Moreover, estimate and plot the effective number of particles $n_{eff,j}$ for $j = 0, 1, \dots, J$ for different values of M.

d)Repeat part c) but with adaptive resampling (i.e., SI–adaptive–R).