LV 11.4500 – UBUNG 8

U9.1 a) In this exercise we will study the total variation of a single Wiener path over the time interval $[0,1]$ by computing

$$TotVar(\ell) := \sum_{k=0}^{2^\ell - 1} |W_{t^\ell_{k+1}} - W_{t^\ell_k}| \qquad \text{for } \ell 1, 2, \ldots, 20$$

on a nested sequence of meshes

$$\{t^1_k\}_{k=0}^{2} \subset \{t^2_k\}_{k=0}^{2^2} \subset \ldots \{t^{20}_k\}_{k=0}^{2^{20}}$$

with $t^\ell_k := k2^{-\ell}$.

Hint: first generate the values of the Wiener path on the fine mesh $\{W_{t^{20}_k}\}_{k=1}^{2^{20}}$ compute $TotVar(20)$. Thereafter use the relation

$$\{W_{t^{\ell-1}_k}\}_{k=0}^{2^{\ell-1}} = \{W_{t^\ell_{2k}}\}_{k=0}^{2^{\ell-1}}$$

as aid to to compute $TotVar(\ell - 1)$.

Verify experimentally for one path that $TotVar(\ell) \approx c2^{\beta\ell}$ for some $\beta > 0$, and determine approximately the value of β from your numerical simulation.

b) for the same Wiener path, verify numerically that

$$QuadraticVar(\ell) := \sum_{k=0}^{2^\ell - 1} |W_{t^\ell_{k+1}} - W_{t^\ell_k}|^2 \to 1 \quad \text{as}$$

as $\ell \to \infty$.

U9.2 The Ornstein-Uhlenbeck equation is defined by

$$dX_t = -\theta X_t dt + \sigma dW_t \tag{1}$$

with constants $\theta, \sigma > 0$.

a) In order to solve (1), we first need to extend Ito's forumla from functions treated in Lecture 19 to those on the form $f(t, X_t)$. That is, using the formal rules

$$(dW)^2 = dt, \quad dWdt = dtdW = 0, \quad (dt)^2 = 0,$$

show that for $f \in C^2(\mathbb{R}^2, \mathbb{R})$ and the SDE (1), Ito's formula leads to

$$df(t, X_t) = \Big(f_t(t, X_t) - \theta X_t f_x(t, X_t) + \frac{\sigma^2}{2} f_{xx}(t, X_t)\Big)dt + \sigma f_x(t, X_t) dW_t.$$

b) Show that

$$X_t = e^{-\theta t} X_0 + \sigma \int_0^t e^{\theta(s-t)} dW_s$$

Hint: Use the integrating factor $e^{\theta t}$ in (1) and apply Ito's formula to $f(t, X_t) = e^{\theta t} X_t$.

c) Show that

$$\int_0^t e^{\theta(s-t)} dW_s \sim N(0, \Sigma_t)$$

and determine Σ_t.

Hint: For some $\Delta t > 0$ consider a partition of $[0,t]$ defined by $t_{k+1} = t_k + e^{2\theta(t-t_k)}\Delta t$. Use the limit definition of Ito integrals in combination with the central limit theorem to verify gaussianity.

d) Show that for any fixed t, the solution of the SDE can be written

$$X_t = A_t X_0 + \xi_t$$

and determine A_t and the distribution of ξ_t.

e) Derive the Fokker–Planck equation for (1) and show that $\tilde{p}(x) \propto \exp(-\theta x^2/\sigma^2)$ is a stationary solution.

U9.3 a) Consider the filtering problem

$$V_{\tau_{j+1}} = \Psi(V_{\tau_j}) := V_{\tau_j} - \frac{1}{4}\int_0^{\Delta\tau} V_{\tau_j+t} dt + \frac{1}{4}\int_0^{\Delta\tau} dW_t^{(j)}$$
$$Y_{\tau_{j+1}} = V_{\tau_{j+1}} + \eta_{j+1}$$

with $V_0 = 1$, $\Delta\tau = 1/2$ and iid $\eta_j \sim N(0, \Gamma)$ (and the standard additional independence $\sigma(\{\eta_j\}_j) \perp \sigma(\{W_s^{(1)}\}_{s\leq\Delta\tau}) \perp \sigma(\{W_s^{(2)}\}_{s\leq\Delta\tau}) \perp \ldots$).

To familiarize yourself with EnKF+Euler–Maruyama filtering, repeat the numerical tests in Lecture 20: generate an observation sequence for $y_{\tau_{1:J}}$ for $J = 10$ from synthetic data $v^\dagger_{\tau_{1:J}}$ for $\Gamma = 1$ and $\Gamma = 1/1000$ and Ψ^N as numerical integrator with timestep $\Delta t = \Delta\tau/N$ for different values of N and M.

b) Consider next the following filtering problem with unknown state and drift coefficient in the model:

$$V_{\tau_{j+1}} = \Psi_\theta(V_{\tau_j}) = V_{\tau_j} - \theta\int_0^{\Delta\tau} V_{\tau_j+t} dt + \frac{1}{4}\int_0^{\Delta\tau} dW_t^{(j)}$$
$$Y_{\tau_{j+1}} = V_{\tau_{j+1}} + \eta_{j+1}.$$

with $V_0 = 4$, $\Delta\tau = 1/10$ and observations over the time frame $[0, 10]$, iid observation noise $\eta_j \sim N(0, 0.01)$ and the prior $\theta \sim U(0,1)$.

Task: 1. Set $N = 10$ and generate synthetic data $y_{\tau_{1:J}}$ from dynamics $v^\dagger_{\tau_{1:J}}$ with parameter $\theta^\dagger = 2/3$ for $J = 100$ and 2. implement a filtering strategy for recovering the true model value $\theta^\dagger$ conditional on the observation sequence $y_{\tau_{1:J}}$ and the prior for θ. Study the robustness of the method when varying the ensemble size M.

U9.4 Consider applying EnKF+Euler–Maruyama method to a filtering problem with SDE dynamics. Under the assumption of sufficient regularity and a finite observation sequence $J < \infty$ over a finite time interval $[0, T]$, how should one split the computational budget between the degrees of freedom M and N? (Where M is the ensemble size and N is the timestep parameter defined through $\Delta t = \Delta\tau/N$.)

U9.5 Describe how to combine particle filtering with Euler–Maruyama to solve filtering problems with SDE dynamics and discrete-time observations.